Course

MTH-5150-2

Optimization in a General Context

Mathematics

INTRODUCTION

The goal of the *Optimization in a General Context* course is to enable adult learners to deal with optimization situations that involve using graphs or linear programming, or finding measurements in order to design or use three-dimensional objects in a general context.

In this course, adult learners are introduced to linear programming and are required to apply their knowledge of arithmetic and algebra to different situational problems involving specific constraints. They use their ability to translate a situation into equations or inequalities and to work with algebraic expressions. They draw the Cartesian coordinate graph of the corresponding system, interpreting the inequality relations. In optimization situations, they determine the values of the decision variables in the function that optimizes (minimizes or maximizes) a situation involving a number of constraints, which in fact represent limitations related to real-life situations involving optimization.

Adults also learn how to use graphs to model situational problems involving optimization. The situations may involve project planning, communication or distribution networks, circuits, incompatibilities, localizations, strategies and so on. Depending on the situation, adult learners use different types of graphs: trees, directed or undirected, coloured or not coloured, weighted or unweighted. To optimize certain situations, they find the critical path, use graph colouring, or determine trees of minimum value or the shortest path. In addition, they can use graphs to represent or construct labyrinths or winning strategy games. Using a graph representing the outcome of a winning strategy game, adult learners can work backwards to determine which positions could lead to victory.

In this course, adult learners explore situational problems that involve finding certain measurements related to congruent, similar or equivalent figures as well as the properties of figures and metric or trigonometric relations. They compare equivalent figures and determine which one is most appropriate to meet certain objectives (e.g. to maximize or minimize space). They analyze and interpret situations involving measuring instruments, photography, lamps and shadows, etc. For instance, in situations involving packaging, adult learners determine the most economical shape for a container of a given volume, taking into account factors such as ease of storage. They may calculate the ratio between the volume and total area and see a connection between the value of the ratio and the most economical shape.

By the end of this course, adult learners will be able to use half-planes, weighted and directed graphs, or similar, congruent or equivalent figures to represent concrete situations. They will produce clear and accurate work in accordance with the rules and conventions of mathematics. Optimizing situations using systems of first-degree inequalities, inference functions (graphs) or calculations involving geometric data will enable them to make decisions. In addition, they will use different registers of representation to generalize results and extend them to other situations.

SUBJECT-SPECIFIC COMPETENCIES

In order to solve the situational problems in this course, adult learners will use the following three subject-specific competencies:

- *Uses strategies to solve situational problems*
- *Uses mathematical reasoning*
- *Communicates by using mathematical language*

The use of effective strategies involves employing rigorous mathematical reasoning and communicating clearly by observing the codes and conventions of mathematical language. Adult learners solve situational problems by using all three subject-specific competencies and other resources.

The following section explains how to use the three subject-specific competencies to solve a situational problem.

PROCESS AND STRATEGIES

To solve a situational problem, adult learners need effective strategies that they can adapt to the situations at hand.

Adult learners solve situational problems using a four-phase process:

- **representation**
- **planning**
- **activation**
- **reflection**

The following table gives an overview of the phases in the problem-solving process, as well as a few examples of strategies adult learners can use in dealing with various situations. These phases are not necessarily carried out in the order indicated above. Adult learners may have to go back and forth among the four phases in order to solve a situational problem.

PROCESS AND STRATEGIES	
REPRESENTATION	
- Adult learners examine the situational problem to identify the context, the problem and the task to be performed. They use observational and representational strategies that are essential to inductive reasoning. - In attempting to understand the context and the problem, they use deductive reasoning. - Certain observations can be made in representing situations involving geometric optimization (e.g. a cube is the right prism with the largest volume).	
Examples of strategies	• Determining the type of optimization most suited to the situation by asking questions • Using vertices and edges, making an intuitive sketch of a graph that represents the problem • Listing the mathematical concepts related to graph theory when working with a problem that involves finding the optimal path • Describing the characteristics of the situation • Gathering relevant information (e.g. vertices, edges, circuit) • Determining the nature of the task to be carried out (instructions, expected results, goal, time allotted, etc.) based on specifications, a scale drawing or literal descriptions • Writing literal expressions to represent the elements of the situation that seem relevant, thus making it easier to find measurements or spatial representation
PLANNING	
- In planning their solution, adult learners look for ways of approaching the problem and choose those that seem the most efficient and economical. - At this stage, they are able to express the constraints of the situation in mathematical language. - Everything they do is aimed at finding optimal solutions. For example, when attempting to find the optimal path in a graph or a tree, they use their intuition to highlight the edges that could represent this path. - They may use a graphic representation of the situation to highlight certain metric or trigonometric relations.	
Examples of strategies	• Breaking down the situational problem into subproblems (finding the optimal path involves breaking down the graphs into circuits and paths) • Finding an algebraic rule that reflects the best relationship between the constraints and possible consequences of the situational problem: determining the relevant parameters of the scanning line or the economic function • Breaking down the situational problem into subproblems in order to find a measurement using the metric relations in similar figures
ACTIVATION	
- When dealing with a situational problem, adult learners deduce the scale of the axes by analyzing the maximum and minimum values of the variables in order to graph the half-planes resulting from the constraints. They also use simple substitution to deduce certain values of the points of intersection of the boundary lines. - To avoid confusion, they use the symbols, terms and notation in accordance with their meaning. - They distinguish between the different types of figures by making sure that the proper codes and rules have been observed. - They take into account the proportion indicated and use the symbols and conventions related to the concept in question. - They use a diagram or sketch to illustrate their proof, thereby making it easier for the reader to understand.	
Examples of strategies	• Proceeding by trial and error to mathematize certain constraints or to identify the different paths in the graph • Enumerating all the possible paths in a graph in order to choose the best solution • Constructing tables of values in order to find two points to represent the boundary lines of the polygon of constraints • Trying to identify the figures that optimize the situation in order to fully understand, among other things, the relationship between the characteristics (area and volume) of an object and the constraints that affect the space it occupies
REFLECTION	

PROCESS AND STRATEGIES	
- Adult learners use a reflective approach throughout the situation and always review the phases in the problem-solving process and the choices made, with a view to validating the solution. - By reviewing the steps in their work, they hone their ability to use exact mathematical language, particularly when producing a message. They must ensure that their message is clear and that they have observed the relevant codes and conventions. - They make conjectures about particular or special cases involving any triangle in order to validate certain results using reasoning.	
Examples of strategies	• Checking their solution by, for example, comparing the number of possible solutions for a system of equations with the number of solutions found, or using their intuition to make sure that the coordinates of the points they have found are those of the vertices of the polygon of constraints • Distinguishing the strategies that are useful in linear programming from those that are used in graph theory • Determining the strategies for dealing with situational problems in geometry (e.g. applying a rule, referring to a geometry principle, using a formula)

CROSS-CURRICULAR COMPETENCIES

Cross-curricular competencies are not developed in a vacuum; they are rooted in situational problems. To varying degrees, the cross-curricular competencies contribute to the development of the subject-specific competencies, and vice versa.

Several cross-curricular competencies can be useful in dealing with the family of learning situations *Optimizing solutions*. Two of these are considered particularly relevant to this course: *Communicates appropriately* and *Exercises critical judgment*.

Communication-Related Competency

There are many situational problems involving optimization (e.g. planning a market study to predict company earnings, optimizing the expenses involved in planning an advertising campaign, choosing the most economical type of packaging). By developing the competency *Communicates appropriately*, adult learners will be able to approach a problem in a manner that is not purely mathematical. Since communication is an interactive process that demands constant adjustment to a range of possible meanings and mutual expectations, linear programming alone is not sufficient to solve a problem. In addition, knowledge of the methods for producing and distributing media products and the ability to use various techniques, technologies and modes of communication are assets that go beyond the scope of mathematics.

Intellectual Competency

The cross-curricular competency *Exercises critical judgment* can be very useful in a learning situation that involves planning the creation of a media product. In these situations, adult learners become aware of the need to respect intellectual property, defend freedom of expression and respect the privacy and reputation of others. This focus is broader than an approach limited to the mathematization of constraints and the optimization of objective functions. It can also encourage adult learners to overcome prejudice and to go beyond intuitive assumptions. In a learning situation that involves designing packaging, adult learners may be called upon to *exercise their critical judgment* with regard to the sometimes excessive use of packaging in marketing. They therefore become aware that responsible choices made by consumers can save money and energy and, above all, promote better environmental management. Other situations may also enable adult learners to understand the issues surrounding the use of space in different fields, such as advertising or the arts, or in different types of facilities.

SUBJECT-SPECIFIC CONTENT

In this course, adult learners use and build on their previously acquired knowledge of arithmetic and algebra. This knowledge is useful for taking into account constraints in an optimization context. In order to deal effectively with situational problems, they will add to what they have learned by mastering the mathematical knowledge specific to this course.

Prescribed Knowledge

In order to deal effectively with the learning situations in this course, adult learners develop the following three integrative processes:

- **optimizing a situation using linear programming**
- **optimizing a situation using graph theory**
- **optimizing space when designing or using three-dimensional objects**

These processes, which are applied in the learning situations in this course, foster the integration of mathematical knowledge and the subject-specific competencies. The learning situations must involve at least one of these integrative processes. However, there must be a sufficient variety of learning situations to cover all three processes. The learning situations may be purely mathematical or based on everyday events.

Mathematical Knowledge	Restrictions and Clarifications
Algebraic expressions	
• Solving first-degree inequalities in two variables	

<table>
<tr><th>Mathematical Knowledge</th><th>Restrictions and Clarifications</th></tr>
<tr><td>Linear programming</td><td></td></tr>
<tr><td>System of first-degree inequalities in two variables</td><td></td></tr>
<tr><td>Representing the constraints and the function to be optimized (objective or economic function)</td><td>Constraints can be represented algebraically or graphically.

In this course, the function to be optimized is expressed solely as an equation of the form $Ax + By + C = Z$, where A, B and C are rational numbers.</td></tr>
<tr><td>Determining and interpreting the vertices and the feasible region (bounded or unbounded)</td><td></td></tr>
<tr><td>Changing the conditions associated with the situation to provide a more optimal solution</td><td></td></tr>
<tr><td>Graph</td><td></td></tr>
<tr><td>Representing and modelling a situation using a graph</td><td></td></tr>
<tr><td>Comparing different graphs</td><td>The graphs studied in this course, including trees, are the following:
simple graphs (vertices and edges only)
directed graphs
coloured graphs
weighted graphs
connected graphs
complete graphs
The different elements related to the graphs studied in this course are the following: vertex, edge, loop, degree of a vertex, distance, path, circuit, simple path and simple circuit.</td></tr>
<tr><td>Finding Euler and Hamiltonian paths and circuits, a critical path, the shortest path, a tree of minimum or maximum values or the chromatic number</td><td></td></tr>
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Mathematical Knowledge	Restrictions and Clarifications
Finding measurements	
• Equivalent figures	
• Finding measurements: ○ positions ○ angles ○ lengths (segments, chords) ○ areas ○ volumes	These measurements are found by applying the properties of congruent, similar or equivalent figures as well as the properties of figures and trigonometric relations. Metric relations may also be applied.
• Relations in triangles	The trigonometric relations studied involve the cosine law. The trigonometric ratios in right triangles, the cosine law and Hero's formula may also be applied.

Principles
Adult learners must master the following compulsory principles, which may be used in a proof: **P13.** A connected graph contains a Euler path if and only if the number of vertices with an odd-numbered degree is 0 or 2. **P14.** A connected graph contains a Euler circuit if and only if the degrees of all its vertices are even numbers. **P15.** The chromatic number of a graph is less than or equal to $r + 1$, where r is the largest degree of its vertices. **P16.** Regular polygons have the smallest perimeter of all equivalent polygons with n sides. **P17.** Of two equivalent regular convex polygons, the polygon with the most sides will have the smaller perimeter. (Ultimately, an equivalent circle will have the smaller perimeter.) **P18.** Cubes have the largest volume of all rectangular prisms with the same total surface area. **P19.** Spheres have the largest volume of all solids with the same total surface area. **P20.** Cubes have the smallest total surface area of all rectangular prisms with the same volume. **P21.** Spheres have the smallest total surface area of all solids with the same volume. **P22.** The square of the length of a side of any triangle is equal to the sum of the squares of the lengths of the other two sides, minus twice the product of the lengths of the other two sides multiplied by the cosine of the contained angle (*cosine law*).

Cultural References

For many years, business and government decision-makers have had to solve problems involving combinatorial analysis, random experiments and corporate competitiveness. This is why many mathematicians have studied the issue.

Swiss mathematician Leonhard Euler (1707-1783), a pioneer in pure and applied mathematics, is regarded as the author of the first theorem stemming from graph theory. Linear programming, which is a branch of optimization commonly used in decision-making, originated with the work of French mathematician Joseph Fourrier (1768-1830) on systems of inequalities, even though these systems have been attributed to American mathematician George Dantzig (1914-2005). While in the United States Air Force during the Second World War, Dantzig developed a technique for solving the army's logistical problems at a minimum cost. This technique, which combines power and flexibility, was soon adopted in business and industry. Businesses used it to solve major economic problems, while industry applied it to production management.

Since the 1970s, linear programming has been applied in a variety of fields such as health care, the environment, agriculture, communications, the oil industry, chemistry, computer science, energy, transportation, industrial production and finance. This breakthrough is the result of advancements in computer technology, which made it possible to deal with situations involving an astronomical number of calculations. Examples given during the course will enable adult learners to understand the importance of linear programming.

Graph theory is another tool that has been used to make transportation and freight companies more profitable. Adult learners who are interested in learning more about the applications of this theory could contact one or more transportation companies to understand how they use graphs to save time and money when determining their routes.

Since prehistoric times, people have used packaging techniques to preserve and transport food. First, they used animal hides, leaves and shells and then began to make packaging: amphora, jars, baskets, and glass and metal containers. With the industrial revolution (end of the 19th century and beginning of the 20th century), throwaway containers made their appearance. Packaging also acquired some additional functions—namely, to make storage easier and inform the consumer. Packaging designers must meet industry demands at the least cost by finding measurements to solve the optimization problems submitted to them.

FAMILY OF LEARNING SITUATIONS

The situations in the family *Optimizing solutions* involve problems that can be solved in part through optimization using linear programming or graph theory or by finding measurements. The *Optimization in a General Context* course provides adult learners with an opportunity to learn how to maximize a profit, a process or a number of objects or people, and to minimize costs or losses.

In the situational problems in this course, adult learners list the mathematical concepts related to graph theory when working with a problem that involves finding the optimal path, use their intuition to highlight the edges that could represent the optimal path in a graph or a tree, and go back to the problem to check whether the solution is closely related to the vertices or the boundary of the polygon of constraints. In a geometrical context, they must distinguish between the different types of figures by making sure that the proper mathematical rules and codes have been observed.

BROAD AREAS OF LEARNING

The broad areas of learning deal with major contemporary issues. Ideally, the situations to be studied should be selected in keeping with the educational aims of the broad areas of learning, which provide the situational problems with contexts that make the learning process meaningful. Two broad areas of learning are considered particularly relevant to this course: Media Literacy, and Environmental Awareness and Consumer Rights and Responsibilities.

Media Literacy

Some learning situations can involve examining the role the media plays in marketing a product in order to help adult learners become more aware and exercise their critical, ethical and aesthetic judgment with respect to the media. For example, adult learners could try to optimize an investment in an advertising campaign, while taking into account the constraints of the problem. Such factors as the gender, age and income of the target group must be taken into account in mathematizing these constraints. This type of situation ties in with the educational aim of this broad area of learning.

Environmental Awareness and Consumer Rights and Responsibilities

Some of the learning situations encountered in this course could lead adult learners to examine the use of plastics in packaging. For example, they could compare the amount of packaging used by volume depending on whether the product is sold individually or in a family-sized package. They might also consider studying a larger population. In addition, they could try to determine whether there is a relationship between the size and shape of the packaging, the amount of space containers occupy in refrigerated display cases and the cost of transportation. This exercise is aimed at making them aware of the impact of daily consumer choices so that they can develop a more active relationship with their environment. It also enables them to maintain a critical attitude toward consumption and the exploitation of the environment, which ties in with the educational aim of this broad area of learning. In other situations, adult learners may have to organize a company’s delivery schedule. In doing so, they may assess the cost of fuel based on different routes, using graph theory, and thereby determine the possible cost savings and benefits for the environment.

Example of a learning situation

All learning situations and situational problems, regardless of the broad area of learning to which they are related, require the active participation of the adult learner. They provide an opportunity to develop the targeted subject-specific and cross-curricular competencies, to acquire mathematical concepts and to mobilize a variety of useful resources.

The table below presents the elements needed to develop a learning situation or situational problem. It specifies these elements for the situational problem described on the following page.

ELEMENTS NEEDED TO DEVELOP A LEARNING SITUATION OR A SITUATIONAL PROBLEM	
Targeted broad area of learning • Helps contextualize learning and makes it meaningful.	• Media Literacy
Prescribed subject-specific competencies • Are developed through the active participation of adult learners.	• Uses strategies to solve situational problems • Uses mathematical reasoning • Communicates by using mathematical language
Prescribed family of learning situations • Consists of real-life situations applicable to a given course. • Helps adult learners acquire mathematical knowledge.	• Optimizing solutions
Targeted cross-curricular competencies • Are developed at the same time and in the same context as the subject-specific competencies.	• Communicates appropriately • Exercises critical judgment
Prescribed essential knowledge • Refers to knowledge to be applied and concepts to be acquired.	• See list

This section provides an example of a situational problem along with possible tasks involved in its mathematical processing. The context can be used as a common thread throughout the learning situation. The learning activities are not spelled out; rather, the focus is on a relevant example of mathematical processing using the four phases in the problem-solving process: representation, planning, activation and reflection. Although not explicitly stated, the elements of the situational problem identified in the previous table, i.e. the broad area of learning, subject-specific competencies, family of learning situations, cross-curricular competencies and essential knowledge, can be discerned, and must form a coherent and meaningful whole.

Teachers may choose to use any of these elements as objects of learning. For instance, learning can focus on actions associated with the phases in the problem-solving process, actions related to the subject-specific or cross-curricular competencies, or actions related to the prescribed knowledge. Teachers can also use the example provided to construct other complex tasks or learning activities related to the mathematical knowledge adult learners must acquire.

Situational problem	Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the *Optimizing solutions* family of learning situations
A company wants to launch a new product. The marketing director must draw up a budget to promote the product in the media. The first step is to develop an advertising plan. Naturally, the director wants the most media coverage at the least cost. Once the most suitable medium has been chosen, adult learners are asked to choose the least expensive advertising plan from among four possibilities.	**Integrative process:** *Optimizing a situation using linear programming* In carrying out the four phases in the problem-solving process, adult learners could: Representation • Determine the key elements to be considered: the number of days required to complete each step (design and implementation) and the cost of each one • Indicate the obstacles to be overcome in implementing the plan Planning • Refer to the solution of a similar situational problem in drawing up the plan • Determine the mathematical knowledge needed to deal with the situation: identify the variables, determine the constraints, establish a system of first-degree inequalities in two variables

Situational problem	Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the *Optimizing solutions* family of learning situations	
To optimize the advertising plan, adult learners must first analyze the four possibilities submitted and graph the constraints associated with each plan (e.g. the minimum number of employees required and their hourly wage, the maximum cost of the materials or the cost of designing the plan).	Activation	• Mathematize the constraints (e.g. the price charged by each supplier) using inequalities • Draw a polygon of constraints to represent the situation and then optimize the expenses • Determine the vertex of the polygon of constraints that represents the lowest cost • Calculate the cost associated with this vertex
	Reflection	• Compare their solution and results with those of others in order to identify the strengths and weaknesses of the model(s), etc. • Determine the number of suppliers required to ensure that the model is reliable and realistic • Make sure the solution is realistic • Examine how a change in one of the constraints will determine which supplier provides the best plan (e.g. Will an increase in the minimum wage result in a different optimal solution?) • Indicate what change a company must make to its plan in order to obtain the contract

END-OF-COURSE OUTCOMES

To solve situational problems in the family of learning situations *Optimizing solutions*, adult learners optimize a situation using linear programming or graph theory, or for the purpose of designing or using three-dimensional objects. To do this, they use the three subject-specific competencies, *Uses strategies to solve situational problems*, *Uses mathematical reasoning* and *Communicates by using mathematical language*.

To use linear programming to solve a situational problem involving optimization, adult learners decode the elements of mathematical language, use symbols and mathematical rules to represent the constraints of the situation in the form of a system of inequalities, produce a graph to illustrate the polygon of constraints, and determine the coordinates of the vertices. Then they examine every possible solution using the scanning line and distinguish between discrete and continuous solutions. Lastly, they take the time to validate their solution according to the context, and distinguish the vertices that belong to or limit inequalities. Faced with a conjecture, they compare, evaluate and critically examine choices or processes and establish proofs, if applicable. After taking a position, they choose the best solution. They justify all the steps in their process (solution and result) and determine either an optimal solution or reasons for rejecting the conjecture. They also explain the possible effects of changing certain constraints and generalize the situations, as needed.

By using graph theory to solve situational problems involving optimization, adult learners can clearly represent the situation using a graph, identify the vertices and edges that correspond to the context, and judge whether or not to assign a value or direction to the edges. They count the possible number of paths and select the critical path by analyzing their solution and comparing it with the context of the situational problem. In addition, when proving principles related to graphs, they use the three theorems covered in the course to find results through deductive or inductive reasoning.

In exploring situational problems involving the optimization of space when designing or using three-dimensional objects, adult learners compare equivalent figures and determine which ones best meet certain objectives (maximize or minimize the space). They establish relationships between the total surface areas and volume of solids. They employ various strategies to find an algebraic solution to the situational problem, applying their mathematical knowledge related to functions. They use their knowledge of geometry to design and produce plans and objects. They determine different measurements based on established definitions, properties, formulas or principles in the case of triangles, plane figures, or congruent, similar or equivalent solids. In addition, when producing and validating their solution, they rigorously justify the different steps in their work, providing formal proofs.

Throughout the problem-solving process, adult learners apply their mathematical knowledge (linear programming, graphs and finding measurements). Their use of symbols, terms and notation related to this knowledge is accurate, and they always refer to different sources to validate the laws, theorems, corollaries or lemmas they deduce or induce so that they can improve their mathematical literacy. In addition, they do not hesitate to ask for help when they encounter difficulties.

EVALUATION CRITERIA FOR THE COMPETENCIES TARGETED BY THE COURSE

Uses strategies to solve situational problems

- *Indication (oral or written) that the situational problem has been understood*
- *Application of strategies and appropriate mathematical knowledge*
- *Development of an appropriate solution**
- *Appropriate validation of the steps** in the solution*

* The solution includes a procedure, strategies and a final answer.
** The mathematical model, operations, properties or relations involved.

Uses mathematical reasoning

- *Formulation of a conjecture suited to the situation*
- *Correct use of appropriate mathematical concepts and processes*
- *Proper implementation of mathematical reasoning suited to the situation*
- *Proper organization of the steps in an appropriate procedure*
- *Correct justification of the steps in an appropriate procedure*

Communicates by using mathematical language

- *Correct interpretation of a mathematical message*
- *Production of a message in keeping with the terminology, notation and conventions of mathematics, and suited to the context*