Course

MTH-5164-2

Sequences and Series in an Applied Context

Mathematics

INTRODUCTION

The goal of the course *Sequences and Series in an Applied Context* is to enable adult learners to deal effectively with situations that involve using number sequences to represent a dependency relationship between quantities in an applied context.

The concept of arithmetic sequences is no doubt one of the cornerstones of modern mathematics, and its study provides a simpler representation of the direct relationship between two variables. When scientists study observable phenomena for the first time, their only resources are numbers that they compile in tables or graphs. Searching for a dependency relationship between the numbers can become a seemingly never ending task. A return to basic arithmetic often provides a better understanding of how variables are related. This course involves studying strategies that will make it possible to identify patterns that reflect a close relationship between variables. Different registers of representation (ordered or numbered sequences, tables of values, graphs or algebraic rules) can be used in the search for patterns.

This course presents different methods and strategies in order to help adult learners to distinguish between arithmetic and geometric sequences recursively and explicitly; to establish a direct relationship between geometric sequences and exponential functions; to deduce and apply formulas for the general term and the sum of a geometric sequence.

In addition, adult learners further explore mathematical formalism through the use of symbols for summation (Σ), logical connectors and set notation, in order to make algebraic manipulations less onerous. The concept of limit is introduced in an intuitive manner in order to determine the convergence of sequences and series.

At the end of this course, adult learners will be able to apply their knowledge of first- or second-degree polynomial functions, exponential functions and logarithmic functions, and to solve situational problems involving arithmetic or geometric sequences and series in accordance with the symbols and conventions of mathematics. By algebraically or graphically representing a situation using real functions, they will induce results through interpolation or extrapolation. They can interpolate or extrapolate using a table of values, a graph or algebra when the algebraic rule is given. Lastly, adult learners will use different registers of representation (tables of values, graphs or algebraic rules) to generalize a model so that it can be applied to a range of situations described by sequences and series.

SUBJECT-SPECIFIC COMPETENCIES

In order to solve the situational problems in this course, adult learners will use the following three subject-specific competencies:

- *Uses strategies to solve situational problems*
- *Uses mathematical reasoning*
- *Communicates by using mathematical language*

The use of effective strategies involves employing rigorous mathematical reasoning and communicating clearly while observing the codes and conventions of mathematical language. Adult learners solve situational problems by using all three subject-specific competencies and other resources.

The following section explains how to use the three subject-specific competencies to solve a situational problem.

PROCESS AND STRATEGIES

To solve a situational problem, adult learners need effective strategies that they can adapt to the situations at hand.

Adult learners solve situational problems using a four-phase process:

- **representation**
- **planning**
- **activation**
- **reflection**

The following table gives an overview of the phases in the problem-solving process, as well as a few examples of strategies adult learners can use in dealing with various situations. These phases are not necessarily carried out in the order indicated above. Adult learners may have to go back and forth among the four phases in order to solve a situational problem.

PROCESS AND STRATEGIES	
REPRESENTATION	
- Adult learners examine the situational problem to identify the context, the problem and the task to be performed. They use observational and representational strategies that are essential to inductive reasoning. - They increase their knowledge of mathematical notation and symbols related to functions and inverse functions expressed in the general form.	
Examples of strategies	• Writing literal expressions to represent the elements of the situation that seem relevant, thus making it easier to identify a dependency relationship and determine the variables in the situation • Using examples involving numbers, determining the types of relationship that exist between the variables in the situation • Exploring a geometric sequence recursively
PLANNING	
- In planning their solution, adult learners look for ways of approaching the problem and choose those that seem the most efficient. - They attempt to extrapolate results using an algebraic rule or a graph, thus expanding their networks of cognitive resources. - To correctly plan their solution, they decode elements of mathematical language, such as the meaning of the symbols, terms and notation used, as well as the different registers of representation.	
Examples of strategies	• Drawing a concept map showing the different steps in the solution • Referring to a list of elements to be considered in consolidating their work plan (e.g. the scale of the axes, the increasing and decreasing intervals, the maximum or the minimum, if any) • Generating the first three terms in a geometric sequence • Writing an exponential function given the corresponding geometric sequence
ACTIVATION	
- When dealing with a situational problem, adult learners use reasoning to establish structured and functional relationships among different aspects of their knowledge, thus expanding their networks of mathematical cognitive resources. - They use different strategies by associating pictures, objects or concepts with mathematical terms and symbols and by switching from one register of representation to another.	
Examples of strategies	• Changing perspective • Systematically determining the general form of the algebraic rule of a function • Finding combinations in order to determine the rule of a quadratic function • Creating a linear model from a non-linear one by replacing the values of the independent (X) or dependent (Y) variable, or both, with their logarithm • Comparing sequences with known algebraic models • Selecting the algebraic function that is best suited to the situation by determining the differences between the terms in the sequence and the differences in the *y*-values of different functions
REFLECTION	
- Adult learners use a reflective approach throughout the situational problem and always review the steps in the problem-solving process and the choices made, with a view to validating the solution. - Through reasoning, they can make conjectures about particular or special cases to validate certain results. - They use different strategies to make sure that the dependent and independent variables are properly defined, that the axes are correctly scaled, that no unit of measure has been omitted and that the data have been correctly transcribed.	
Examples of strategies	• Checking their solution by, for example, making sure that the resulting values satisfy the range of the function, or substituting the values of the variables in the algebraic expression in order to validate a graphical interpolation or extrapolation • Exploring the limits of a formula deduced from the general term • Validating their solution by means of a graph of the algebraic rule • Determining the difference between the values in the situational problem and the corresponding model

CROSS-CURRICULAR COMPETENCIES

Cross-curricular competencies are not developed in a vacuum; they are rooted in situational problems. To varying degrees, the cross-curricular competencies contribute to the development of the subject-specific competencies, and vice versa.

Several cross-curricular competencies can be useful in dealing with the family of learning situations Relationship between quantities. Two cross-curricular competencies are considered particularly relevant to this course: *Adopts effective work methods* and *Communicates appropriately*.

Methodological Competency

It is easier to interpret a situation if it is represented by a numerical sequence or series. Adult learners develop the competency *Adopts effective work methods* when they are required to analyze data derived from compiled observations (e.g. observations related to a complex biological phenomenon for which it is not easy to identify the variables involved).

Communication-Related Competency

The need to make extrapolations or provide proof and justifications could motivate adult learners to develop the competency *Communicates appropriately*. Providing proof requires that adult learners organize their thinking, formulate arguments using the correct vocabulary, show respect for others and be open to their ideas.

SUBJECT-SPECIFIC CONTENT

In this course, adult learners use and build on their previously acquired knowledge of arithmetic and algebra. In order to deal effectively with situational problems, they will add to what they have learned by mastering the mathematical knowledge specific to this course.

Prescribed Knowledge

In order to deal effectively with the learning situations in this course, adult learners develop the following three integrative processes:

- **using a sequence or series to represent a situation**
- **interpolating or extrapolating from a numerical or graphical model**
- **using an algebraic or graphical model of a function to generalize a set of situations**

These processes, which are applied in the learning situations in this course, foster the integration of mathematical knowledge and the subject-specific competencies. The learning situations must involve at least one of these integrative processes. However, there must be a sufficient variety of learning situations to cover all three processes.

Mathematical Knowledge	Restrictions and Clarifications

Arithmetic and geometric sequences
• Determining the general term, the convergence and the limits of a sequence
• Experimenting, observing, interpreting, describing and representing situations using number sequences
	In this course, the study of arithmetic and geometric sequences is limited to:
	• $u_{n+1} = u_n + d$ (arithmetic sequence with common difference d)
	    o e.g.: sequence of odd numbers
	• $u_{n+1} = qu_n$ (geometric sequence)
	    o e.g.: Sierpinski triangle
	
	The characteristics of the sequences studied in this course are:
	• increasing
	• decreasing
	• strictly increasing
	• strictly decreasing
	• monotonic
	• bounded from above
	• bounded from below
	• bounded
	
	In this course, the following statements are studied to determine the convergence (limit) of sequences.
	• All monotonic sequences are bounded and convergent
	• All convergent sequences are bounded
	• If $\{u_n\}$ is a sequence converging toward *a* and $\{v_n\}$ is a sequence converging toward *b*, then:
	    ○ $\{u_n + v_n\}$ converges toward $a + b$
	    ○ $\{u_n \cdot v_n\}$ converges toward $a \cdot b$
	    ○ $\left\{\frac{u_n}{v_n}\right\}$ converges toward $\frac{a}{b}$
	    ○ $\{\lambda u_n\}$ converges toward λa

Mathematical Knowledge	Restrictions and Clarifications
Series • Determining the formula, convergence and limits of a series • Experimenting, observing, interpreting, describing and representing situations using number series	In this course, the following tests and statement are studied to determine the convergence of series. • Comparison test Given U = $\sum_{i=1}^{\infty} u_i$ and V = $\sum_{i=1}^{\infty} v_i$, series with positive terms: V is converging $\wedge\ u_i \leq v_i, \forall i \Rightarrow$ U is converging V is diverging $\wedge\ u_i \geq v_i, \forall i \Rightarrow$ U is diverging • Quotient test Given U = $\sum_{i=1}^{\infty} u_i$ and $L = \lim_{n\to\infty} \left\| \frac{u_{n+1}}{u_n} \right\|$: o $L < 1 \Rightarrow$ U converges absolutely o $L > 1 \Rightarrow$ U diverges absolutely o $L = 1 \Rightarrow$ no conclusion can be drawn • A series is said to converge absolutely or diverge absolutely when the series of the absolute values of the terms is convergent or divergent respectively.

Cultural References

Sequences of real numbers have always been linked to experimental mathematics. Sequences and series were used in ancient civilizations, for instance, when Archimedes developed his recurrence algorithm, whose successive application provides an approximation of areas and a value for the constant π (pi). Sequences and series were also used in Egypt to compute a square root using Heron of Alexandria's method.

During WWII, advances in the field of informatics rekindled interest in sequences and series. Examples of these advances include the invention of a calculator that had SIN, COS, TAN, LOG and EXP keys which were programmed using the developments around convergent series.

FAMILY OF LEARNING SITUATIONS

The situations in the family *Relationship between quantities* involve problems that can be solved in part by using an algebraic or graphical model of a function to represent a relationship between quantities. The *Sequences and Series in an Applied Context* course provides adult learners with an opportunity to learn how to express a connection or a dependency relationship between quantities.

In the situational problems in this course, adult learners become more familiar with the mathematical symbols and notation related to sequences and series. In addition to extrapolating results using a function or a graph, they use a scale appropriate to the context so that the graph they draw in solving the situational problem makes sense in the context.

BROAD AREAS OF LEARNING

The broad areas of learning deal with major contemporary issues. Ideally, the situations to be studied should be selected in keeping with the educational aims of the broad areas of learning, which provide the situational problems with contexts that make the learning process meaningful. Two broad areas of learning are considered particularly relevant to this course: Citizenship and Community Life and Career Planning and Entrepreneurship.

Citizenship and Community Life

Many situations that involve a tenuous balance among several observable parameters highlight problems that have direct and indirect consequences on our economic life. Take, for example the problem of field mouse infestations, which wine producers know only too well. In order to familiarize themselves with certain phenomena stemming from the interplay of the parameters in this complex situation, adult learners could study these parameters by using sequences and series to simplify analysis and better identify the real-world implications that go far beyond the mathematics involved. This problem illustrates the close correlation between economic performance and the control of the reproduction of a species and the regulations that need to be enacted to deal with the situation. This ties in directly with the life of their society and is consistent with the educational aim of this broad area of learning.

Career Planning and Entrepreneurship

Adult learners faced with a learning situation that involves sequences and series used in biology or other branches of science and that also involves multiple variables may be required to determine the change in the birth rate of a species at the expense of another or to find a way of better controlling the prey-predator dynamic that could affect the balance of species. For example, effective management of the field mouse infestation problem could have positive economic spinoffs since a decrease in the number of field mice will lead to an increase in the harvests and therefore an increase in the related stock price. This ties in with one of the focuses of development of this broad area of learning, which deals with the exploration of plans for the future based on their interests and aptitudes.

EXAMPLE OF A LEARNING SITUATION

All learning situations and situational problems, regardless of the broad area of learning to which they are related, require the active participation of the adult learner. They provide an opportunity to develop the targeted subject-specific and cross-curricular competencies, to acquire mathematical concepts and to mobilize a variety of useful resources.

The table below presents the elements needed to develop a learning situation or situational problem. It specifies these elements for the situational problem described on the following page.

ELEMENTS NEEDED TO DEVELOP A LEARNING SITUATION OR A SITUATIONAL PROBLEM	
Targeted broad area of learning – Helps contextualize learning and makes it meaningful.	• Career Planning and Entrepreneurship
Prescribed subject-specific competencies – Are developed through the active participation of adult learners.	• Uses strategies to solve situational problems • Uses mathematical reasoning • Communicates by using mathematical language
Prescribed family of learning situations – Consists of real-life situations applicable to a given course. – Helps adult learners acquire mathematical knowledge.	• Relationship between quantities
Targeted cross-curricular competencies – Are developed at the same time and in the same context as the subject-specific competencies.	• Adopts effective work methods • Communicates appropriately
Prescribed essential knowledge – Refers to mathematical knowledge and concepts to be acquired.	• See list

This section provides an example of a situational problem along with possible tasks involved in its mathematical processing. The context can be used as a common thread throughout the learning situation. The learning activities are not spelled out; rather, the focus is on a relevant example of mathematical processing using the four phases in the problem-solving process: representation, planning, activation and reflection. Although not explicitly stated, the elements of the situational problem identified in the previous table, i.e. the broad area of learning, subject-specific competencies, family of learning situations, cross-curricular competencies and essential knowledge, can be discerned, and must form a coherent and meaningful whole.

Teachers may choose to use any of these elements as objects of learning. For instance, learning can focus on actions associated with the phases in the problem-solving process, actions related to the subject-specific or cross-curricular competencies, or actions related to the prescribed knowledge. Teachers can also use the example provided to construct other complex tasks or learning activities related to the mathematical knowledge adult learners must acquire.

Situational problem	Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the ***Relationship between quantities*** family of learning situations
For a number of years, wine growers in the region have carried out a daily campaign against the field mouse infestation. These little rodents attack the vines and can destroy an entire crop in a single season. Several ecological and sustainable methods have been implemented, including bringing birds of prey into the region in order to stabilize the prey-predator ratio. To form a more accurate picture of the situation, wine growers have asked experts to carry out a study.	**Integrative process:** *Using an algebraic or graphical model of a function to generalize a set of situations* In carrying out the four phases in the problem-solving process, adult learners could: Representation • Select the relevant information (e.g. number of field mice counted in a given period, times at which measurements were taken, length of observation periods, temperature) and disregard superfluous information • List the different modes of representation that are best suited to address the situation Planning • Organize the information • Study the changes in the first and second elements in the sequence • Find a relationship between pairs of experimental parameters • List the elements needed to draw the graph

Situational problem	Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the ***Relationship between quantities*** family of learning situations	
Using the data that ecologists have compiled, the adult learner produces a report describing the situation.	Activation	• Make a table of the data related to the situation, taking into account the limitations and precision of the measuring instruments used • Study the first and second differences in order to highlight the arithmetic or geometric nature of the sequence • Match an exponential function with a geometric sequence or a linear function with an arithmetic sequence or a quadratic function with the sum of the data values • Compare the coefficients of determination for each function • Choose the most appropriate one
	Reflection	• State the different conclusions associated with the different algebraic models • Highlight the differences between the *y*-values predicted by the functions and the *y*-values obtained through experimentation • Describe future scenarios using examples

END-OF-COURSE OUTCOMES

To solve situational problems in the family of learning situations *Relationship between quantities*, adult learners represent a situation, carry out interpolations and extrapolations and use an algebraic or graphical model to generalize a set of situations. To do this, they use the three subject-specific competencies, *Uses strategies to solve situational problems*, *Uses mathematical reasoning* and *Communicates by using mathematical language*.

When studying a situation using sequences and series, adult learners represent the situation by making a graph of the arithmetic data, which allows them to efficiently choose the type of function that best describes the situation. They combine different registers of representation as needed to produce a message in accordance with the notation, rules and conventions of mathematical language. They use problem-solving strategies to make comparisons, propose corrections, present favourable or optimal solutions, or issue recommendations. They formulate constructive criticism and make informed decisions concerning issues in a variety of fields, including technical fields (e.g. graphics, biology, physics, administration).

In interpolating or extrapolating results from an algebraic or graphical model, adult learners use their knowledge of different types of functions and strategies, combining reasoning and creativity to overcome obstacles and make decisions. They use structured deductive reasoning and become familiar with the codified form required for their proof. They use the properties of sequences and series to support their argument. They use illustrations, explanations or justifications to describe their conclusions.

To generalize a set of situations using an algebraic or graphical model derived from the preliminary study of sequences and series, adult learners specify the purpose of their communication and switch from one register to another as needed. They demonstrate their understanding of the mathematical concepts in question using a wide range of communication strategies, which enables them to take new requirements into account. They learn and correctly use language that appropriately combines, mathematical, technical and scientific and everyday terms. They deduce new algebraic rules by combining different operations on functions they have mastered. In addition, they use the parameters of the functions effectively to illustrate generalities about a set of functions.

Throughout the problem-solving process, adult learners make an effort to apply their mathematical knowledge (arithmetic and geometric sequences and series, functions, inverse and operations on functions). They make accurate use of symbols, terms and notation related to this knowledge, and they always refer to different sources to validate the laws, theorems, corollaries or lemmas they deduce or induce so that they can improve their mathematical literacy. In addition, they do not hesitate to ask for help when they encounter difficulties.

EVALUATION CRITERIA FOR THE COMPETENCIES TARGETED BY THE COURSE

Uses strategies to solve situational problems

- *Indication (oral or written) that the situational problem has been understood*
- *Application of strategies and appropriate mathematical knowledge*
- *Development of an appropriate solution**
- *Appropriate validation of the steps** in the solution*

* The solution includes a procedure, strategies and a final answer.
** The mathematical model, operations, properties or relations involved.

Uses mathematical reasoning

- *Formulation of a conjecture suited to the situation*
- *Correct use of appropriate mathematical concepts and processes*
- *Proper implementation of mathematical reasoning suited to the situation*
- *Proper organization of the steps in an appropriate procedure*
- *Correct justification of the steps in an appropriate procedure*

Communicates by using mathematical language

- *Correct interpretation of a mathematical message*
- *Production of a message in keeping with the terminology, rules and conventions of mathematics, and suited to the context*