Course

MTH-5173-2

Geometric Representation in a Fundamental Context 2

Mathematics

INTRODUCTION

The goal of the *Geometric Representation in a Fundamental Context 2* course is to enable adult learners to deal with situations that involve describing and graphically representing geometric loci in a fundamental context.

Adult learners enhance their knowledge of the relationships between geometry and algebra by using trigonometric identities and studying conics, among other things. With respect to trigonometry, they use their understanding of equivalence relations and their ability to work with algebraic expressions to prove identities involving trigonometric expressions and to solve trigonometric equations. In studying conics, they discover other applications, in particular with regard to telecommunications systems. They examine conics based on a cross-section of a cone or through various hands-on activities (folding, play of light and shadows, construction). They observe patterns and attempt to define the different conics. They determine the equations associated with them and describe each region using inequalities. They find the coordinates of points of intersection and different measurements using algebra and changing variables when necessary.

In learning about the concept of vector, adult learners build on what they learned in the previous cycle. Vectors make it possible to take a new approach to certain situations involving geometry and can be related to different concepts such as proportionality, linear functions, first-degree equations and geometric transformations associated with motion. Adult learners can then compare the properties of real numbers with those of vectors. When performing vector operations, they use the Chasles relation, among other things. Depending on the situations involved, adult learners can also work with different linear combinations or determine the coordinates of a point of division using the product of a vector and a scalar. Vectors are studied in both the Euclidean and the Cartesian planes.

By the end of this course, adult learners will be able to use trigonometric relations, the properties of equivalent figures and metric relations in circles to describe and represent geometric transformations. Used with matrices, analytic geometry makes it possible to algebraically model certain geometric transformations of objects. Furthermore, adult learners will be able to describe, represent and generalize certain characteristics of geometric loci in the Cartesian plane by using vectors in accordance with the mathematical rules and conventions used in geometry.

SUBJECT-SPECIFIC COMPETENCIES

In order to solve the situational problems in this course, adult learners will use the following three subject-specific competencies:

- *Uses strategies to solve situational problems*
- *Uses mathematical reasoning*
- *Communicates by using mathematical language*

The use of effective strategies involves employing rigorous mathematical reasoning and communicating clearly by observing the codes and conventions of mathematical language. Adult learners solve situational problems by using all three subject-specific competencies and other resources.

The following section explains how to use the three subject-specific competencies to solve a situational problem.

PROCESS AND STRATEGIES

To solve a situational problem, adult learners need effective strategies that they can adapt to the situations at hand.

Adult learners solve situational problems using a four-phase process:

- **representation**
- **planning**
- **activation**
- **reflection**

The following table gives an overview of the phases in the problem-solving process, as well as a few examples of strategies adult learners can use in dealing with various situations. These phases are not necessarily carried out in the order indicated above. Adult learners may have to go back and forth among the four phases in order to solve a situational problem.

PROCESS AND STRATEGIES	
REPRESENTATION	
- Adult learners examine the situational problem to identify the context, the problem and the task to be performed. They use strategies that are essential to inductive reasoning. - They distinguish between the mathematical and everyday meanings of the terms used so that they can understand what is meant by focus, ellipse, vertex, arc, etc.	
Examples of strategies	• Using an estimate, a scale plan or a literal description to determine the nature of the task involved • Illustrating their understanding of the situational problem by drawing on relevant mathematical knowledge related to conics • Describing the situation in their own words in order to show that they understand the situational problem, and comparing their understanding of it with that of their classmates or teacher
PLANNING	
- To plan their solution, adult learners look for ways of approaching the problem and choose those that seem most efficient. - They use different registers of representation to illustrate certain properties of geometric transformations. - They make connections between the elements of a message, for example, to determine if the algebraic rule accurately reflects the situation.	
Examples of strategies	• Dividing the situational problem into subproblems, for example, in order to find a measurement using the metric relations in a circle • Using lists, tables, diagrams, concrete materials or drawings to plan their solution
ACTIVATION	
- In dealing with a situational problem, adult learners use mathematical reasoning to prove geometry principles pertaining to metric relations in a circle, giving several examples before drawing conclusions. - They use exact mathematical language to ensure that the various steps involved in working out the solution are carried out efficiently.	
Examples of strategies	• Using the characteristics of a conic, making a sketch to predict results • Simplifying the situational problem by comparing it with a similar problem that has already been solved • Predicting the possible solutions for a system of second-degree equations involving conics to be able to understand, for example, the relationship between the degree of an equation and the maximum number of possible solutions
REFLECTION	
- Adult learners use a reflective approach throughout the situation and always review the phases in the problem-solving process and the choices made, with a view to validating the solution. - Reasoning helps them make conjectures about particular or special cases involving any triangle in order to validate certain relationships. - They consult different reference documents to validate their mathematical message when they use new mathematical symbols to describe the organization or representation of their physical environment by means of conics and vectors.	
Examples of strategies	• Checking their solution by means of examples or counterexamples • Determining the strategies for dealing with situational problems in geometry (e.g. making a drawing, changing perspective) • Using a calculator or geometric modelling software to validate their work.

CROSS-CURRICULAR COMPETENCIES

Cross-curricular competencies are not developed in a vacuum; they are rooted in situational problems. To varying degrees, the cross-curricular competencies contribute to the development of the subject-specific competencies, and vice versa.

Several cross-curricular competencies can be useful in dealing with the family of learning situations *Measurement and spatial representation*. Two of these are considered relevant to this course: *Uses information and communications technologies* and *Solves problems*.

Methodological Competency

Simulation makes it considerably easier to understand optical phenomena. The use of specialized geometry and virtual laboratory software simplifies the physical representation of different phenomena. Adult learners could also use a spreadsheet program to perform different calculations involving trigonometric ratios. In developing the competency *Uses information and communications technologies*, adult learners could come to regard specialized software as a valuable tool for representing reality.

Intellectual Competency

Analyzing problems with a view to explaining the mathematical phenomena in different situations helps adult learners develop the competency *Solves problems*. If warranted by the degree of complexity involved, adult learners use strategies to examine and select the key elements needed to solve the problem. They make appropriate use of the mathematical concepts they have learned (e.g. conics and vectors). It should be noted that dealing with a situation involves exploring avenues that may not lead to the correct solution. Because of the variety of situations they encounter, adult learners are able to hone their problem-solving skills.

SUBJECT-SPECIFIC CONTENT

In this course, adult learners use and build on their previously acquired knowledge of geometry and algebra. In order to deal effectively with situational problems, they will add to what they have learned by mastering the mathematical knowledge specific to this course.

Prescribed Knowledge

In order to deal effectively with the learning situations in this course, adult learners develop the following two integrative processes:

- **describing geometric loci and representing them graphically**
- **using vectors to generalize geometry principles**

These processes, which are applied in the learning situations in this course, foster the integration of mathematical knowledge and the subject-specific competencies. The learning situations must involve at least one of these integrative processes. However, there must be a sufficient variety of learning situations to cover both processes.

Mathematical Knowledge	Restrictions and Clarifications
Geometric transformations	
• Representing and interpreting geometric transformations	Geometric transformations are represented and interpreted using algebraic rules.
Finding measurements	
• Equivalent figures • Finding measurements of: o arcs or angles o lengths (segments, chords) o areas o volumes o capacities	These measurements are found by applying the properties of congruent, similar or equivalent figures, as well as metric or trigonometric relations.
Geometric loci: conics	
• Describing, representing and constructing a conic	Conics are the only geometric loci studied in this option. The conics studied are: • the parabola (centred at the origin and translated) • the circle (centred at the origin) • the ellipse (centred at the origin) • the hyperbole (centred at the origin)

Mathematical Knowledge	Restrictions and Clarifications
Geometric loci: conics (cont.)	The elements described are: • the radius • the axes • the directrix • the vertices • the foci • the asymptotes • the regions *Conics are described using only the standard form of the algebraic rule.*
• Solving a system of second-degree equations with respect to conics	
• Determining the coordinates of points of intersection between a line and a conic or between a parabola and another conic	*Inequalities related to conics are studied in this option.*
Trigonometric relations	
• Standard unit circle (radian and arc length)	
• Manipulating simple trigonometric expressions using definitions (sine, cosine, tangent, secant, cosecant and cotangent)	*Only the Pythagorean identities and the properties of periodicity and symmetry are studied in this course.* *Formulas for finding the sum or difference of two angles are studied in this option.*

Mathematical Knowledge	Restrictions and Clarifications
Vectors	
• Resultant and projection	
• Operations on vectors	*Vector* refers to a geometric or free vector. Operations on vectors are limited to the following: • adding and subtracting vectors • multiplying a vector by a scalar • the scalar product of two vectors • the properties of the scalar product of two vectors: o commutativity of the scalar product $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$ o distributivity over vector addition $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$ o associativity of scalars $k_1\vec{u} \bullet k_2\vec{v} = k_1k_2(\vec{u} \bullet \vec{v})$ • linear combination • properties of vectors o associativity $k_1(k_2\vec{u}) = (k_1k_2)\vec{u}$ o existence of a scalar acting as an identity element $1\vec{u} = \vec{u}$ o existence of a zero scalar and an absorbing element: $\left(k\vec{u} = \vec{0}\right) \Leftrightarrow \left(k = 0 \vee \vec{u} = \vec{0}\right)$ o distributivity over vector addition $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ o distributivity over scalar addition $(k_1 + k_2)\vec{u} = k_1\vec{u} + k_2\vec{u}$
• Determining the coordinates of a point of division	The coordinates of a point of division are determined using the product of a vector and a scalar.

Principles

Adult learners must master the following compulsory principles, which may be used in a proof:

P17. Regular polygons have the smallest perimeter of all equivalent polygons with *n* sides.

P18. Of two equivalent convex polygons, the polygon with the most sides will have the smaller perimeter. (Ultimately, an equivalent circle will have the smaller perimeter.)

P19. Cubes have the largest volume of all rectangular prisms with the same total surface area.

P20. Spheres have the largest volume of all solids with the same total surface area.

P21. Cubes have the smallest total surface area of all rectangular prisms with the same volume.

Vectors

- Given $\vec{u}$, $\vec{v}$, and $\vec{w}$, which are vectors in the plane, as well as scalars **r** and **s**,

P22. $\left(\mathbf{r}\vec{u} = \vec{0}\right) \Leftrightarrow \left(r = 0 \vee \vec{u} = \vec{0}\right)$

P23. If $\vec{u}$ and $\vec{v}$ are non-collinear vectors, then $(\mathbf{r}\vec{u} = \mathbf{s}\vec{v}) \Leftrightarrow (\mathbf{r} = \mathbf{s} = 0)$

P24. ($\vec{w}$ is collinear at $\vec{u}$) $\Leftrightarrow$ $(\exists! \, \mathbf{r} \in \mathbb{R} : \vec{w} = \mathbf{r}\vec{u})$

P25. ($\vec{u}$ and $\vec{v}$ are non-collinear) $\Leftrightarrow$ $(\forall \vec{w}, \ \exists! \, \mathbf{r} \in \mathbb{R}, \ \exists! \, \mathbf{s} \in \mathbb{R} : \vec{w} = \mathbf{r}\vec{u} + \mathbf{s}\vec{v})$

P26. $(\vec{u} \perp \vec{v}) \Leftrightarrow (\vec{u} \bullet \vec{v}) = 0$

Cultural References

Before becoming a science in its own right, classical mechanics was a branch of mathematics. Up until the end of the 18th century, mechanics was used as a field of application to test mathematical laws and theories. Mechanics and mathematics are still closely linked today. The need to model experimental data prompted mathematicians to develop theories related to geometry or differential equations. Many important mathematicians, such as Euler, Cauchy and Lagrange, made often decisive contributions in this area.

In this course, adult learners could also explore three-dimensional representation. The space between the eyes makes humans capable of spatial visualization of solids. Both eyes do not see the exact same image of an object. The brain processes these differences and makes it possible not only to construct a 3-D object, but also to gauge the distance between the object and a person. This is the idea behind 3-D IMAX films, as the distance between the two cameras used is equivalent to the average distance between human eyes. The use of hypersterography to create topographical maps is based on the same principle. When aerial surveys are conducted, two photographs of the same place are taken at different times (and therefore from different points of view). Binocular vision can be simulated from these two photographs. However, since the distance between the two images is greater than the space between the eyes, the resulting 3-D effect is exaggerated (hence the prefix *hyper-*), thereby making it possible to create more accurate topographical maps. Students could carry out a project in which they represent a three-dimensional object to help them assimilate the related mathematical concepts.

FAMILY OF LEARNING SITUATIONS

The situations in the family *Measurement and spatial representation* involve problems that can be solved in part through the mathematical description or representation of geometric transformations of objects or geometric loci. The *Geometric Representation in a Fundamental Context 2* provides adult learners with an opportunity to develop their spatial representation skills.

In the situational problems in this course, adult learners examine conics based on a cross-section of a cone or through various hands-on activities, find the coordinates of points of intersection and different measurements using algebra and changing variables where necessary, and compare the properties of real numbers with those of vectors.

BROAD AREAS OF LEARNING

The broad areas of learning deal with major contemporary issues. Ideally, the situations to be studied should be selected in keeping with the educational aims of the broad areas of learning, which provide the situational problems with contexts that make the learning process meaningful. Two broad areas of learning are considered particularly relevant to this course: Health and Well-Being, and Citizenship and Community Life.

Health and Well-Being

Many people must wear corrective lenses because they are either shortsighted or farsighted, but how are vision problems corrected? Starting with this question, adult learners can explore the basic concepts of optics as they relate to vision, including mirrors and light. Many mathematical concepts can be used to explain these phenomena. For example, adult learners could use their knowledge of conics to understand the concept of the focus of a lens (e.g. magnifying glass) or a parabolic mirror. They could also do research on the possible applications of the principles of conics in everyday technology (e.g. rotating lights on emergency vehicles). The study of optics can help adult learners adopt a safer lifestyle because they will have a better understanding of how light waves work and will therefore be able to use them without jeopardizing their health, which ties in with the focuses of development of this broad area of learning.

Citizenship and Community Life

Games have been an integral part of societies throughout history. Some games help develop social skills, while others call more upon individual qualities (e.g. games that test thinking abilities, intelligence, strength and dexterity). The latter provide useful material for the study of mathematics, particularly those related to kinematics and conics. Adult learners could be asked to calculate the parabolic trajectory of a projectile (e.g. javelin, dart), or the velocity resulting from the impact between two objects in games or sports such as billiards, curling or other, more ancient games. The study of games and sports over time and in different places helps adult learners become open to the world, which ties in with the educational aim of this broad area of learning.

EXAMPLE OF A LEARNING SITUATION

All learning situations and situational problems, regardless of the broad area of learning to which they are related, require the active participation of the adult learner. They provide an opportunity to develop the targeted subject-specific and cross-curricular competencies, to acquire mathematical concepts and to mobilize a variety of useful resources.

The table below presents the elements needed to develop a learning situation or situational problem. It specifies these elements for the situational problem described on the following page.

ELEMENTS NEEDED TO DEVELOP A LEARNING SITUATION OR A SITUATIONAL PROBLEM	
Targeted broad area of learning – Helps contextualize learning and makes it meaningful.	• Health and Well-Being
Prescribed subject-specific competencies – Are developed through the active participation of adult learners.	• Uses strategies to solve situational problems • Uses mathematical reasoning • Communicates by using mathematical language
Prescribed family of learning situations – Consists of real-life situations applicable to a given course. – Helps adult learners acquire mathematical knowledge.	• Measurement and spatial representation
Targeted cross-curricular competencies – Are developed at the same time and in the same context as the subject-specific competencies.	• Uses information and communications technologies • Solves problems
Prescribed essential knowledge – Refers to mathematical knowledge and concepts to be acquired.	• See list

This section provides an example of a situational problem along with possible tasks involved in its mathematical processing. The context can be used as a common thread throughout the learning situation. The learning activities are not spelled out; rather, the focus is on a relevant example of mathematical processing using the four phases in the problem-solving process: representation, planning, activation and reflection. Although not explicitly stated, the elements of the situational problem identified in the previous table, i.e. the broad area of learning, subject-specific competencies, family of learning situations, cross-curricular competencies and essential knowledge, can be discerned, and must form a coherent and meaningful whole.

Teachers may choose to use any of these elements as objects of learning. For instance, learning can focus on actions associated with the phases in the problem-solving process, actions related to the subject-specific or cross-curricular competencies, or actions related to the prescribed knowledge. Teachers can also use the example provided to construct other complex tasks or learning activities related to the mathematical knowledge adult learners must acquire.

Situational problem	Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the *Measurement and spatial representation* family of learning situations
The headlights on a car have two light bulbs and a concave mirror that acts as a reflector. At a science fair, an adult learner decides to present an exhibit illustrating the operating principle of these headlights and to explain the difference between high beams and low beams. In addition to having to define concepts such as the radius of curvature or the focus, the adult learner must draw graphs to illustrate how the position of the light bulb affects the amount of space illuminated in front of the car and must also use appropriate mathematical language in the process.	**Integrative process:** *Describing geometric loci and representing them graphically* In carrying out the four phases in the problem-solving process, adult learners could: Representation - Determine the key elements in this situation (e.g. the radius of curvature of the mirror, the position of the light bulbs with respect to the mirror) - Draw a diagram of the situation by sketching the light rays emitted by the bulbs and reflected by the curved mirror Planning - Look for possible solutions through brainstorming - Use concrete materials to work out the solution

Situational problem	Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the *Measurement and spatial representation* family of learning situations	
Adult learners will use concepts related to optics, but this learning situation could be studied in science courses at the same time.	Activation	- Draw on the mathematical knowledge needed to deal with the situation: line tangent to a circle, perpendicular lines, normal, measures of angles, etc. - Predict how the light rays are likely to behave in the two situations, then use examples to check this proposition - Draw a graph showing how the behaviour of the light rays will change, depending on the position of the light source
	Reflection	- Compare their solution and results with those of their classmates or teacher to determine whether the theoretical models are appropriate - Examine the effect of changing a parameter such as the radius of curvature or the position of the focus

END-OF-COURSE OUTCOMES

To solve situational problems in the family of learning situations *Measurement and spatial representation*, adult learners describe geometric loci, represent them graphically or algebraically, and use vectors to generalize geometry principles. To do this, they use the three subject-specific competencies, *Uses strategies to solve situational problems*, *Uses mathematical reasoning* and *Communicates by using mathematical language*.

To describe geometric loci and represent them graphically or algebraically, adult learners use different mathematical models and different types of strategies, combining reasoning and creativity to overcome obstacles. They use structured deductive reasoning and become familiar with the codified form required for their proof. They illustrate, explain or justify their arguments. They use different types of proofs and different lines of reasoning, including proof by exhaustion. The latter is used in particular in analyzing or conducting case studies, or in applying a generalization process leading to the validation of a conjecture. Adult learners observe specific real-life cases and generalize their observations. They analyze data in order to identify the necessary and sufficient conditions for drawing a conclusion, make decisions and determine how best to approach, optimize or adjust a situation. In addition, they base their reasoning on a Euclidean or Cartesian plane in order to determine measurements, optimize distances, construct geometric loci, represent the relative positions of figures or justify recommendations.

To prove a theorem using vectors, adult learners translate the hypotheses and thesis using vectors and construct an equality. They develop the equality and use the Chasles relation to reduce the initial equality to its simplest form. As needed, they apply the properties of the scalar product of two vectors. They make connections between vector notation and the properties of geometric figures. In addition, they justify all the steps in their solution, applying the properties of vectors so as to communicate clearly and concisely.

Throughout the problem-solving process, adult learners apply their mathematical knowledge (geometric transformations, geometric loci, trigonometric relations and vectors). Their use of symbols, terms and notation related to this knowledge is accurate, and they always refer to different sources to validate the laws, theorems, corollaries or lemmas they deduce or induce so that they can improve their mathematical literacy. In addition, they do not hesitate to ask for help when they encounter difficulties.

EVALUATION CRITERIA FOR THE COMPETENCIES TARGETED BY THE COURSE

Uses strategies to solve situational problems

- *Indication (oral or written) that the situational problem has been understood*
- *Application of strategies and appropriate mathematical knowledge*
- *Development of an appropriate solution**
- *Appropriate validation of the steps** in the solution*

* The solution includes a procedure, strategies and a final answer.

** The mathematical model, operations, properties or relations involved.

Uses mathematical reasoning

- *Formulation of a conjecture suited to the situation*
- *Correct use of appropriate mathematical concepts and processes*
- *Proper implementation of mathematical reasoning suited to the situation*
- *Proper organization of the steps in an appropriate procedure*
- *Correct justification of the steps in an appropriate procedure*

Communicates by using mathematical language

- *Correct interpretation of a mathematical message*
- *Production of a message in keeping with the terminology, rules and conventions of mathematics, and suited to the context*