

Course
MTH-5171-2
Algebraic and Graphical Modelling
in a Fundamental Context 2

Mathematics



INTRODUCTION

The goal of the *Algebraic and Graphical Modelling in a Fundamental Context 2* course is to enable adult learners to deal with situations that involve using an algebraic or graphical model to represent a dependency relationship between quantities in a fundamental context.

In this course, adult learners encounter several situations involving periodic functions. The standard unit circle provides a context for analyzing the specific case of trigonometric functions. This circle enables adult learners to visualize the periodic nature of trigonometric functions and trigonometric lines, to derive properties and to prove certain identities. For example, the use of piecewise functions makes it possible to analyze a variety of situations, such as remuneration during and after regular working hours. A rule is determined and defined for each interval of the domain. The concept of continuity then comes into play and can be used to interpret the variation in the rate of change.

Operations on functions are studied in practical contexts that involve, for example, calculating the total income tax payable (addition) or sales tax (composition). The study of these operations must not be an end in itself, but should be part of the process of analyzing situations and developing corresponding models. The concepts of infinity and continuity allow adult learners to understand the asymptotes of functions, and vice versa. The definition of the concept of limit is introduced intuitively (without referring to symbols) in order to clarify certain situations. The study of rational, tangent, exponential or logarithmic functions can also give rise to a discussion of these concepts.

By the end of this course, adult learners will be able to use algebra to represent concrete situations. They will produce clear and accurate work in accordance with the rules and conventions of mathematics. By algebraically or graphically representing a situation using real functions and their inverse, they will be able to employ inductive or deductive reasoning to obtain results through interpolation or extrapolation. In addition, adult learners will use different registers of representation to generalize results and extend them to other situations.

SUBJECT-SPECIFIC COMPETENCIES

In order to solve the situational problems in this course, adult learners will use the following three subject-specific competencies:

- *Uses strategies to solve situational problems*
- *Uses mathematical reasoning*
- *Communicates by using mathematical language*

The use of effective strategies involves employing rigorous mathematical reasoning and communicating clearly by observing the codes and conventions of mathematical language. Adult learners solve situational problems by using all three subject-specific competencies and other resources.

The following section explains how the three subject-specific competencies are used to solve a situational problem.

PROCESS AND STRATEGIES

To solve a situational problem, adult learners need effective strategies that they can adapt to the situations at hand.

Adult learners solve situational problems using a four-phase process:

- **representation**
- **planning**
- **activation**
- **reflection**

The following table gives an overview of the phases in the problem-solving process, as well as a few examples of strategies adult learners can use in dealing with various situations. These phases are not necessarily carried out in the order indicated above. Adult learners may have to go back and forth among the four phases in order to solve a situational problem.

| PROCESS AND STRATEGIES | |
|---|---|
| REPRESENTATION | |
| <ul style="list-style-type: none"> - Adult learners examine the situational problem to identify the context, the problem and the task to be performed. They use observational and representational strategies that are essential to inductive reasoning. - By becoming more familiar with the symbols and notation used in the course, they develop their ability to communicate using mathematical language. | |
| Examples of strategies | <ul style="list-style-type: none"> • Writing literal expressions to represent the elements of the situation that seem relevant, thus making it easier to identify a dependency relationship and determine the variables in the situation • Using examples involving numbers, determining the type of relationship that exists between the variables of the situation • Sketching a Cartesian coordinate graph to represent the dependency relationship between the variables • Gathering relevant information in order to describe their understanding of the dependency relationship between the variables |
| PLANNING | |
| <ul style="list-style-type: none"> - In planning their solution, adult learners look for ways of approaching the problem and choose those that seem the most efficient. - They attempt to extrapolate results using an algebraic rule or graph, thus expanding their networks of cognitive resources. - To correctly plan their solution, they decode the elements of mathematical language such as the meaning of the symbols, terms and notation used, as well as the different registers of representation. | |
| Examples of strategies | <ul style="list-style-type: none"> • Drawing a concept map showing the different steps in the solution • Listing the elements needed to represent a function graphically or algebraically • Systematically asking questions to consolidate their work plan, for example, What should the scale of the axes be? |
| ACTIVATION | |
| <ul style="list-style-type: none"> - When dealing with a situational problem, adult learners make connections between the algebraic and graphical form in order to derive and generalize the rules. - They use the appropriate scale so that the graph they draw in solving the situational problem makes sense in light of the context. | |
| Examples of strategies | <ul style="list-style-type: none"> • Changing perspective • Systematically determining the general form of the algebraic rule of a function • Finding combinations in order to determine the rule of a quadratic function |
| REFLECTION | |
| <ul style="list-style-type: none"> - Adult learners use a reflective approach throughout the situation and always review the phases in the problem-solving process and the choices made, with a view to validating the solution. - Reasoning helps them make conjectures about particular or special cases in order to validate certain results, and enables them to reject extrapolations that would yield nonsensical results. - They use different strategies to make sure that the dependent and independent variables are properly defined, that the axes are correctly scaled, that no unit of measure has been omitted and that the data have been correctly transcribed. | |

CROSS-CURRICULAR COMPETENCIES

Cross-curricular competencies are not developed in a vacuum; they are rooted in situational problems. To varying degrees, the cross-curricular competencies contribute to the development of the subject-specific competencies, and vice versa.

Several cross-curricular competencies can be useful in dealing with the family of learning situations *Relationship between quantities*. Two of these are considered particularly relevant to this course: *Exercises critical judgment* and *Communicates appropriately*.

Intellectual Competency

Representing a situation by means of an algebraic model could provide adult learners with an opportunity to use the competency *Exercises critical judgment*. Adult learners must be discerning when analyzing demographic data in order to determine a population growth model. They must exclude superfluous data from their analysis and select the functional model that applies to the situation and that will differ according to the population growth pattern in question. Adult learners use their judgment to qualify the importance of the margin of error. They question the future growth trends, knowing that the population cannot increase indefinitely.

Communication-Related Competency

The need to make extrapolations or provide proof and justifications could motivate adult learners to develop the competency *Communicates appropriately*. Providing proof requires that adult learners organize their thinking, formulate arguments using the correct vocabulary, show respect for others and be open to their ideas.

SUBJECT-SPECIFIC CONTENT

In this course, adult learners use and build on their previously acquired knowledge of arithmetic and algebra. In order to deal effectively with situational problems, they will add to what they have learned by mastering the mathematical knowledge specific to this course.

Prescribed Knowledge

In order to deal effectively with the learning situations in this course, adult learners develop the following three integrative processes:

- using an algebraic or graphical model of a function to represent a situation
- interpolating or extrapolating from a graphical model
- using an algebraic or graphical model of a function to generalize a set of situations

These processes, which are applied in the learning situations in this course, foster the integration of mathematical knowledge and the subject-specific competencies. The learning situations must involve at least one of these integrative processes. However, there must be a sufficient variety of learning situations to cover all three processes.

| Mathematical Knowledge | Restrictions and Clarifications |
|---|--|
| <p>Numerical and algebraic expressions</p> <ul style="list-style-type: none"> • Real numbers • Manipulating arithmetic and algebraic expressions | <p>The expressions involve the properties of:</p> <ul style="list-style-type: none"> • absolute values • radicals • exponents • logarithms <p>Working with these expressions enables adult learners to expand their knowledge of the laws of exponents, deduce the different properties of radicals:</p> <ul style="list-style-type: none"> • $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ • $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ • $(\sqrt{a})^2 = a$ • $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$ • $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$ <p>and deduce the following equivalences:</p> <ul style="list-style-type: none"> • $a^b = c \Leftrightarrow \log_a c = b$ • $\log_a c^n = n \log_a c$ • $\log_a c = \frac{\ln c}{\ln a} = \frac{\log c}{\log a}$ |

| Mathematical Knowledge | Restrictions and Clarifications |
|--|--|
| <p>Relation, function and inverse</p> <ul style="list-style-type: none"> Experimenting with real functions and their inverse as well as observing, interpreting describing and representing them <ul style="list-style-type: none"> Operations on functions Finding the rule of a function or its inverse, depending on the context | <p>Functions can be represented:</p> <ul style="list-style-type: none"> verbally using a table of values algebraically graphically <p>The following real functions will be studied in this course:</p> <ul style="list-style-type: none"> exponential $f(x) = ac^{b(x-h)} + k$ logarithmic $f(x) = a\log_c b(x-h) + k$ rational $f(x) = a\left(\frac{1}{b(x-h)}\right) + k$ square root $f(x) = a\sqrt{b(x-h)} + k$ sinusoidal $f(x) = a \sin b(x-h) + k$ $f(x) = a \cos b(x-h) + k$ tangent $f(x) = a \tan b(x-h) + k$ piecewise absolute value $f(x) = a b(x-h) + k$ <p>The concept of inverse is mainly associated with logarithmic, rational, exponential and square root functions.</p> <p>Experimental data are modelled by using curves related to the functional models under study and associating them with scatter plots.</p> <p><i>The four operations are studied in addition to the composition of functions.</i></p> |

| Mathematical Knowledge | Restrictions and Clarifications |
|--|--|
| <p>Relation, function and inverse (cont.)</p> <ul style="list-style-type: none"> • Describing and interpreting the properties of a function • Interpreting the multiplicative and additive parameters • Determining the type of dependency relationship, using the curve of best fit, with or without the help of technology • Solving equations and inequalities in one variable | <p>The properties of real functions covered in this course are:</p> <ul style="list-style-type: none"> • domain and codomain (range) • increasing and decreasing intervals • extrema • sign • x- and y-intercepts <p>The following equations and inequalities will be studied in this course:</p> <ul style="list-style-type: none"> • trigonometric equations and inequalities of the first degree containing a sine, a cosine or a tangent • square root equations and inequalities • rational equations and inequalities • exponential and logarithmic equations and inequalities that involve applying the properties of exponents and logarithms <p>The concepts of arcsine, arccosine and arctangent are studied mainly as inverse operations involved in solving equations or inequalities. The same is true for the concepts of square root and the logarithm introduced in previous courses.</p> <p><i>The study of exponential and logarithmic functions should focus on bases 2, 10 and e.</i></p> |

Cultural References

In modern societies, overall population growth and the increase in the number of people that fall into each age group are among the major factors (urban, social and economic) to be taken into account for different types of planning. It is important to know the number of children who will be attending school in five years, the number of people who will need a doctor and the number of cars that will be on the road in the years to come. In the 19th century, Benjamin Gompertz was one of the first mathematicians to model population trends. His work and mathematical equations are used today to represent numerous progressions that follow a logistic curve. The increase in the sales of new products, the change in the size of tumours, the evolution of epidemics and the growth of a bacterial or animal population are often represented by the Gompertz model.

The study of functions in concrete situations can provide adult learners with the opportunity to study this type of curve since, in reality, exponential growth is not infinite: the curve enters a different phase when limiting factors are taken into account. It is often by understanding such factors that adult learners can determine the real meaning of the relation represented and understand its progression.

FAMILY OF LEARNING SITUATIONS

The situations in the family *Relationship between quantities* involve problems that can be solved in part by using an algebraic or graphical model of a function to represent a relationship between quantities. The *Algebraic and Graphical Modelling in a Fundamental Context 2* course provides adult learners with an opportunity to express a connection or a dependency relationship between quantities.

In the situational problems in this course, adult learners recognize how a change in a parameter affects the graph of a function, use trial and error to determine the algebraic rule of a function, and deduce certain relationships such as the maximum value of a rational function when the values of the abscissas tend toward infinity by approaching the value of the horizontal asymptote.

BROAD AREAS OF LEARNING

The broad areas of learning deal with major contemporary issues. Ideally, the situations to be studied should be selected in keeping with the educational aims of the broad areas of learning, which provide the situational problems with contexts that make the learning process meaningful. Two broad areas of learning are considered particularly relevant to this course: Citizenship and Community Life, and Environmental Awareness and Consumer Rights and Responsibilities.

Citizenship and Community Life

People can find themselves in a crowd in different circumstances: during a public demonstration, during national holidays or even when school lets out. Everyone knows that a crowd does not behave like each individual in it and that the number of people in a crowd cannot be determined exactly. To become familiar with crowd phenomena, adult learners could compare different situations that arise in their centre. They could calculate the time it takes for a certain number of people to evacuate a given space and modify the parameters in order to find a rule that would make it possible to use the available space more efficiently. They could then make extrapolations involving movie theatres or office buildings. Studying crowd behaviour and designing circulation space to meet specific needs could help adult learners understand certain rules of social conduct and give them the opportunity to improve relationships among citizens. The promotion of the rules of social conduct is one of the focuses of development of this broad area of learning.

Environmental Awareness and Consumer Rights and Responsibilities

Some adult learners struggle with their income, expenses, investments or debts. This is why financial institutions offer a variety of investment opportunities or loans with variable interest rates, cumulative loans and loans with fixed terms. How can adult learners make informed choices and learn to better manage their money? Studying the factors involved in calculating interest rates (rate of change, vertex, initial value) and solving the resulting functions can help them better understand the concepts of budget, investment and credit. In this way, they can gain more control over their finances, which ties in with one of the focuses of development of this broad area of learning.

EXAMPLE OF A LEARNING SITUATION

All learning situations and situational problems, regardless of the broad area of learning to which they are related, require the active participation of the adult learner. They provide an opportunity to develop the targeted subject-specific and cross-curricular competencies, to acquire mathematical concepts and to mobilize a variety of useful resources.

The table below presents the elements needed to develop a learning situation or situational problem. It specifies these elements for the situational problem described on the following page.

| ELEMENTS NEEDED TO DEVELOP A LEARNING SITUATION OR A SITUATIONAL PROBLEM | |
|---|---|
| Targeted broad area of learning – Helps contextualize learning and makes it meaningful. | <ul style="list-style-type: none"> • Citizenship and Community Life |
| Prescribed subject-specific competencies – Are developed through the active participation of adult learners. | <ul style="list-style-type: none"> • Uses strategies to solve situational problems • Uses mathematical reasoning • Communicates by using mathematical language |
| Prescribed family of learning situations – Consists of real-life situations applicable to a given course. – Helps adult learners acquire mathematical knowledge. | <ul style="list-style-type: none"> • Relationship between quantities |
| Targeted cross-curricular competencies – Are developed at the same time and in the same context as the subject-specific competencies. | <ul style="list-style-type: none"> • Exercises critical judgment • Communicates appropriately |
| Prescribed essential knowledge – Refers to mathematical knowledge and concepts to be acquired. | <ul style="list-style-type: none"> • See list |

This section provides an example of a situational problem along with possible tasks involved in its mathematical processing. The context can be used as a common thread throughout the learning situation. The learning activities are not spelled out; rather, the focus is on a relevant example of mathematical processing using the four phases in the problem-solving process: representation, planning, activation and reflection. Although not explicitly stated, the elements of the situational problem identified in the previous table, i.e. the broad area of learning, subject-specific competencies, family of learning situations, cross-curricular competencies and essential knowledge, can be discerned, and must form a coherent and meaningful whole.

Teachers may choose to use any of these elements as objects of learning. For instance, learning can focus on actions associated with the phases in the problem-solving process, actions related to the subject-specific or cross-curricular competencies, or actions related to the prescribed knowledge. Teachers can also use the example provided to construct other complex tasks or learning activities related to the mathematical knowledge adult learners must acquire.

| Situational problem | Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the <i>Relationship between quantities</i> family of learning situations |
|--|---|
| <p>Governments consult demographic studies before allowing a new school to be built in a growing city. Specialists also analyze recent demographic data. They derive an equation that makes it possible to estimate, by extrapolation, the size of a future population, with the smallest possible margin of error.</p> <p>Using fictitious demographic data, adult learners are required to determine the equation that could describe the typical change in the size of the population of a growing city and to analyze the influence of the margin of error on the parameters of this type of equation.</p> | <p>Integrative process: <i>Using an algebraic or graphical model to represent a situation</i></p> <p>In carrying out the four phases in the problem-solving process, adult learners could:</p> <p>Representation</p> <ul style="list-style-type: none"> - Describe the situation in their own words by considering the relevant elements - Make conjectures (e.g. that the function grows to infinity) - Exclude superfluous demographic data, such as average family income <p>Planning</p> <ul style="list-style-type: none"> - Refer to similar situations studied previously - List the types of information needed for the graph: population, year or period covered, margin of error - Draw a graph of the situation to be able to better choose the applicable functional model (e.g. quadratic function, exponential function) |

| Situational problem | Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the <i>Relationship between quantities</i> family of learning situations |
|---------------------|---|
| | <p>Activation</p> <ul style="list-style-type: none"> - Draw on the mathematical knowledge needed to deal with the current situation: suggest a probable algebraic rule, then check whether it applies - Use a spreadsheet program to determine the equation in light of the data provided - Decide on a realistic margin of error - Modify the data in accordance with the chosen margin of error - Find the new applicable algebraic rule - Analyze the different algebraic rules to determine which parameters were affected by the margin of error - Clearly formulate the conclusions of this analysis using the appropriate mathematical language <p>Reflection</p> <ul style="list-style-type: none"> - Compare their solution and results with those of their classmates or teacher in order to determine whether the theoretical models are appropriate - Review the model they initially chose - Determine whether the chosen equation would be valid in the long term (concept of infinity) if the population growth remains constant |

END-OF-COURSE OUTCOMES

To solve situational problems in the family of learning situations *Relationship between quantities*, adult learners represent a situation, interpolate or extrapolate, and generalize a set of situations using an algebraic or graphical model. To do this, they use the three subject-specific competencies, *Uses strategies to solve situational problems*, *Uses mathematical reasoning* and *Communicates by using mathematical language*.

To represent a situational problem using an algebraic or graphical model, adult learners describe, symbolize, code, decode, explain or illustrate the information contained in tables of values or algebraic rules. They combine different registers of representation as needed to produce a message in accordance with the notation, rules and conventions associated with mathematical language. They use problem-solving strategies to make comparisons, propose corrections, present favourable or optimal solutions, or issue recommendations. They formulate constructive criticism and make informed decisions about issues in a variety of fields, including technical fields (e.g. graphics, biology, physics, administration).

To interpolate or extrapolate results from an algebraic or graphical model in order to make decisions, adult learners use different types of functional models and strategies, combining reasoning and creativity to overcome obstacles. They use structured deductive reasoning and become familiar with the codified form required for their proof. They illustrate, explain or justify their arguments. They use different types of proofs and different lines of reasoning, including proof by exhaustion. The latter is used in particular in analyzing or conducting case studies, or in applying a generalization process leading to the validation of a conjecture. Adult learners also observe specific real-life cases and generalize their observations.

To generalize a set of situations using an algebraic or graphical model, adult learners specify the purpose of their communication and switch from one register to another as needed. They demonstrate their understanding of the problems in question using a wide range of communication strategies, which enables them to regulate their transmission of a message based on the specific reaction of the audience or to take new requirements into account. They learn and correctly use language that appropriately combines everyday, mathematical, scientific and technical terms. They deduce new algebraic rules by combining the different operations on functions they have mastered, and prove them, justifying all the steps in their procedure. In addition, they make effective use of the parameters of the functions to illustrate generalities about a set of functions.

Throughout the problem-solving process, adult learners apply their mathematical knowledge (relations, functions, inverse and operations on functions). Their use of symbols, terms and notation related to this knowledge is accurate, and they always refer to different sources to validate the laws, theorems, corollaries or lemmas they deduce or induce so that they can improve their mathematical literacy. In addition, they do not hesitate to ask for help when they encounter difficulties.

EVALUATION CRITERIA FOR THE COMPETENCIES TARGETED BY THE COURSE

Uses strategies to solve situational problems

- *Indication (oral or written) that the situational problem has been understood*
- *Application of strategies and appropriate mathematical knowledge*
- *Development of an appropriate solution**
- *Appropriate validation of the steps** in the solution*

* The solution includes a procedure, strategies and a final answer.

** The mathematical model, operations, properties or relations involved.

Uses mathematical reasoning

- *Formulation of a conjecture suited to the situation*
- *Correct use of appropriate mathematical concepts and processes*
- *Proper implementation of mathematical reasoning suited to the situation*
- *Proper organization of the steps in an appropriate procedure*
- *Correct justification of the steps in an appropriate procedure*

Communicates by using mathematical language

- *Correct interpretation of a mathematical message*
- *Production of a message in keeping with the terminology, rules and conventions of mathematics, and suited to the context*