Course

MTH-5161-2

Algebraic and Graphical Modelling in an Applied Context 2

Mathematics

INTRODUCTION

The goal of the *Algebraic and Graphical Modelling in an Applied Context 2* course is to enable adult learners to deal with situations that involve using an algebraic or graphical model to represent a dependency relationship between quantities in an applied context.

In this course, adult learners interpret parameters in different registers of representation. They learn to model certain situations by means of a periodic function. While their exploration of the standard unit circle introduces the concept of the sinusoidal function, it also helps adult learners make a connection between radians and degrees and calculate arc lengths in different units. Only the sinusoidal model, however, is analyzed in all registers. Operations on functions are examined in the context of concrete situations. In addition to learning about the general form and the factored form of second-degree functions, adult learners discover that the factored form $h(x)$ can be obtained by finding the product or the sum of two functions $f(x)$ and $g(x)$. They also learn that a rational function can be obtained by finding the quotient of two polynomial functions. Adult learners continue to analyze situations in which the rate of change varies according to the interval in question. This type of analysis was introduced in Secondary III and IV. Adult learners can use several functional models to describe how two variables behave in a given interval.

By the end of this course, adult learners will be able to use different functions, including the sinusoidal function, to represent concrete situations. They will produce clear and accurate work in accordance with the rules and conventions of mathematics. By algebraically or graphically representing a situation using real functions and operations on real functions, they will be able to employ inductive reasoning to obtain results through interpolation or extrapolation. In addition, they will use different registers of representation to generalize results and extend them to other situations.

SUBJECT-SPECIFIC COMPETENCIES

In order to solve the situational problems in this course, adult learners will use the following three subject-specific competencies:

- *Uses strategies to solve situational problems*
- *Uses mathematical reasoning*
- *Communicates by using mathematical language*

The use of effective strategies involves employing rigorous mathematical reasoning and communicating clearly by observing the codes and conventions of mathematical language. Adult learners solve situational problems by using all three subject-specific competencies and other resources.

The following section explains how to use the three subject-specific competencies to solve a situational problem.

PROCESS AND STRATEGIES

To solve a situational problem, adult learners need effective strategies that they can adapt to the situations at hand.

Adult learners solve situational problems using a four-phase process:

- **representation**
- **planning**
- **activation**
- **reflection**

The following table gives an overview of the phases in the problem-solving process, as well as a few examples of strategies adult learners can use in dealing with various situations. These phases are not necessarily carried out in the order indicated above. Adult learners may have to go back and forth among the four phases in order to solve a situational problem.

<table>
<tr><th colspan="2">PROCESS AND STRATEGIES</th></tr>
<tr><th colspan="2">REPRESENTATION</th></tr>
<tr><td colspan="2">- Adult learners examine the situational problem to identify the context, the problem and the task to be performed. They use observational and representational strategies that are essential to inductive reasoning.
- They become more familiar with the mathematical symbols and notation related to functions and their inverse expressed in their general form.</td></tr>
<tr><td>Examples of strategies</td><td>• Writing literal expressions to represent the elements of the situation that seem relevant, thus making it easier to identify a dependency relationship and determine the variables in the situation
• Using examples involving numbers, determining the type of relationship that exists between the variables in a situation
• Sketching a Cartesian coordinate graph to represent the dependency relationship between the variables
• Making false assumptions to identify an inconsistency or an absurdity to corroborate their perceptions or call them into question</td></tr>
<tr><th colspan="2">PLANNING</th></tr>
<tr><td colspan="2">- Adult learners look for ways of approaching the problem and choose those that seem the most efficient.
- They attempt to extrapolate results using an algebraic rule or a graph, thus expanding their networks of cognitive resources.
- To correctly plan their solution, they decode the meaning of the symbols, terms and notation used, as well as the different registers of representation.</td></tr>
<tr><td>Examples of strategies</td><td>• Drawing a concept map showing the different steps in the solution
• Referring to a list of elements to be considered in consolidating their work plan (e.g. the scale of the axes, the increasing and decreasing intervals, the maximum or the minimum, if any)</td></tr>
<tr><th colspan="2">ACTIVATION</th></tr>
<tr><td colspan="2">- When dealing with a situational problem, adult learners use reasoning to establish organized and functional relationships among different aspects of their knowledge, thus expanding their networks of mathematical cognitive resources.
- They use different strategies by associating pictures, objects or concepts with mathematical terms and symbols and by switching from one register of representation to another.</td></tr>
<tr><td>Examples of strategies</td><td>• Changing perspective
• Systematically determining the general form of the algebraic rule of a function
• Finding combinations in order to determine the rule of a quadratic function</td></tr>
<tr><th colspan="2">REFLECTION</th></tr>
<tr><td colspan="2">- Adult learners use a reflective approach throughout the situation and always review the phases in the problem-solving process and the choices made, with a view to validating the solution.
- Through reasoning, they could make conjectures about particular or special cases to validate certain results.
- They use different strategies to make sure that the dependent and independent variables are properly defined, that the axes are correctly scaled, that no unit of measure has been omitted and that the data have been correctly transcribed.</td></tr>
<tr><td>Examples of strategies</td><td>• Checking their solution by, for example, making sure that the resulting values satisfy the range of the function, or substituting the values of the variables in the algebraic expression in order to validate a graphical interpolation or extrapolation</td></tr>
</table>

CROSS-CURRICULAR COMPETENCIES

Cross-curricular competencies are not developed in a vacuum; they are rooted in situational problems. To varying degrees, the cross-curricular competencies contribute to the development of the subject-specific competencies, and vice versa.

Several cross-curricular competencies can be useful in dealing with the family of learning situations *Relationship between quantities*. Two of these are considered particularly relevant to this course: *Uses information and communications technologies* and *Uses information*.

Methodological Competency

Adult learners who wish to compile and analyze data related to a situation could use computer tools such as a spreadsheet program or graphing software. These tools make it easier to produce graphs and to change or work with parameters in order to carry out simulations and extrapolations. Through the competency *Uses information and communications technologies*, adult learners will realize that the ability to master these technologies will make their work considerably more interesting.

Intellectual Competency

The information in studies on physical and natural phenomena is not necessarily presented in text or table form. Data can be collected through probes, and in this case, it must be organized so that it can be interpreted as accurately as possible in order to generate the required information. Adult learners could therefore learn to use information consisting of raw data. The competency *Uses information* will help them distinguish between data and information, and understand that proper organization makes it possible to correctly interpret a situation.

SUBJECT-SPECIFIC CONTENT

In this course, adult learners use and build on their previously acquired knowledge of arithmetic and algebra. In order to deal effectively with situational problems, they will add to what they have learned by mastering the mathematical knowledge specific to this course.

Prescribed Knowledge

In order to deal effectively with the learning situations in this course, adult learners develop the following three integrative processes:

- **using an algebraic or graphical model of a function to represent a situation**
- **interpolating or extrapolating from a graphical model**
- **using an algebraic or graphical model of a function to generalize a set of situations**

These processes, which are applied in the learning situations in this course, foster the integration of mathematical knowledge and the subject-specific competencies. The learning situations must involve at least one of these integrative processes. However, there must be a sufficient variety of learning situations to cover all three processes.

Mathematical Knowledge	Restrictions and Clarifications
Numerical and algebraic expressions	
• Completing the square	Completing the square is used for factoring second-degree polynomial functions and switching from one way of writing them to another.
• Dividing second-degree polynomials in one or two variables by a first-degree binomial	The polynomials have a maximum of four terms.
Relation, function and inverse	
• Experimenting with real functions and their inverse as well as observing, interpreting, describing and representing them	Functions can be represented: • verbally • using a table of values • algebraically • graphically

Mathematical Knowledge	Restrictions and Clarifications
Relation, function and inverse (cont.)	The real functions studied are: • second-degree polynomial functions (general, standard and factored form) $f(x) = ax^2 + bx + c$ $f(x) = a(x - h)^2 + k$ $f(x) = a(x - x_1)(x - x_2)$ • exponential functions $f(x) = a\,c^{b(x - h)} + k$ • logarithmic functions $f(x) = a\log_c b(x - h) + k$ • rational functions (standard form) $f(x) = a\left(\frac{1}{b(x - h)}\right) + k$ and also of the following form: $f(x) = \frac{ax + b}{cx + d}$ where a, b, c and d $\in \mathbb{R}$ and $cx + d \neq 0$ • square root functions $f(x) = a\sqrt{b(x - h)} + k$ • sinusoidal functions $f(x) = a \sin b(x - h) + k$ and $f(x) = a \cos b(x - h) + k$ • tangent functions $f(x) = a \tan b(x - h) + k$ • greatest integer functions $f(x) = a\,[b(x - h)] + k$ Experimental data are modelled by using curves related to the functional models under study and associating them with scatter plots. The study of exponential and logarithmic functions should focus on bases 2, 10 and e.

<table>
<tr><th>Mathematical Knowledge</th><th>Restrictions and Clarifications</th></tr>
<tr><td>Relation, function and inverse (cont.)</td><td>The concept of inverse is further studied in Secondary V; it is mainly associated with logarithmic, rational, exponential and square root functions.

The second-degree polynomial function was introduced in previous courses and is now studied in standard form. Converting an expression to the factored form involves using the factoring methods studied in Secondary IV.

Converting an expression to the general form involves expanding the standard form of the expression and makes it possible to establish a correspondence between the parameters. In order to switch from the general form to the standard form, adult learners refer to the established correspondences or complete the square.</td></tr>
<tr><td>Operations on functions</td><td>The four operations are studied in addition to the composition of functions.</td></tr>
<tr><td>Describing and interpreting the properties of a function</td><td>The properties of real functions covered in this course are:
domain and codomain (range)
increasing and decreasing intervals
extrema
sign
x- and y-intercepts</td></tr>
<tr><td>Interpreting additive parameters in the different registers of representation</td><td>The registers of representation studied are:
tables of values
rules
graphs</td></tr>
</table>

<table>
<tr><th>Mathematical Knowledge</th><th>Restrictions and Clarifications</th></tr>
<tr><td>Relation, function and inverse (cont.)
Solving equations and inequalities in one variable</td><td>The following equations and inequalities are studied:
trigonometric equations and inequalities of the first degree containing a sine, a cosine or a tangent
second-degree equations and inequalities
square root equations and inequalities
rational equations and inequalities
exponential and logarithmic equations and inequalities that involve applying the properties of exponents and logarithms
The concepts of arcsine, arccosine and arctangent are studied mainly as inverse operations involved in solving equations or inequalities. The same is true for the concepts of square root and the logarithm introduced in previous courses.</td></tr>
<tr><td>System
Finding the graphical solution for situations consisting of systems of equations or inequalities involving different functional models</td><td></td></tr>
</table>

Cultural References

Human beings have always invented instruments to make their lives easier. In developing their mathematical competencies, adult learners may discover, among other things, that modelling is used to design a number of instruments and machines, that mathematical reasoning plays a role in their manufacture and that different registers of graphical representation must be used to operate them.

By studying the evolution of certain modern instruments (e.g. sphygmomanometer used to measure blood pressure, multimeter), adult learners can make connections between algebraic modelling and the use of these instruments in professional or technical occupations in the sciences. For example, they could analyze a digital camera. Using experimentation and graphs, they could study the relationships between the camera’s resolution, format, pixels, size and storage capacity to determine whether these are functional relationships. They attempt to determine the type of function involved, if any.

In addition, adult learners may discover that the search for all sorts of precise measurements has been a constant concern throughout history.

FAMILY OF LEARNING SITUATIONS

The situations in the family *Relationship between quantities* involve problems that can be solved in part by using an algebraic or graphical model of a function to represent a relationship between quantities. The *Algebraic and Graphical Modelling in an Applied Context 2* course provides adult learners with an opportunity to express a connection or a dependency relationship between quantities.

In the situational problems in this course, adult learners become more familiar with the mathematical symbols and notation related to the functions and their inverse expressed in their general form, extrapolate results using an algebraic rule or a graph, and use the appropriate scale so that the graph they draw in solving the situational problem makes sense in light of the context.

BROAD AREAS OF LEARNING

The broad areas of learning deal with major contemporary issues. Ideally, the situations to be studied should be selected in keeping with the educational aims of the broad areas of learning, which provide the situational problems with contexts that make the learning process meaningful. Two broad areas of learning are considered particularly relevant to this course: Environmental Awareness and Consumer Rights and Responsibilities, and Career Planning and Entrepreneurship.

Environmental Awareness and Consumer Rights and Responsibilities

Adult learners interested in natural disasters such as earthquakes could, through a learning situation on this topic, establish a relationship between a logarithmic function and the calculation of the magnitude of an earthquake. They would discover that this data value is related to a continuous logarithmic function rather than an ordinary scale. Because of the logarithmic nature of this phenomenon, when the energy generated by an earthquake varies by a factor of 10, this corresponds to a one-unit change in magnitude. For example, an earthquake with a magnitude of seven on the Richter scale is ten times stronger than an earthquake with a magnitude of six. Adult learners could use this situation to become more knowledgeable about their environment and improve their understanding of certain phenomena, which ties in directly with one of the focuses of development of this broad area of learning.

Career Planning and Entrepreneurship

In a learning situation involving financial mathematics, adult learners could be asked to determine an annual rate of interest and the value of a term deposit for different investment years, given the initial amount invested and its value ten years later. This situation enables adult learners to use their knowledge of exponential functions to develop a more practical understanding of this function, while learning about the principles of saving. In this way, they could develop strategies that will be useful in carrying out a personal plan, which ties in directly with one of the focuses of development of this broad area of learning.

EXAMPLE OF A LEARNING SITUATION

All learning situations and situational problems, regardless of the broad area of learning to which they are related, require the active participation of the adult learner. They provide an opportunity to develop the targeted subject-specific and cross-curricular competencies, to acquire mathematical concepts and to mobilize a variety of useful resources.

The table below presents the elements needed to develop a learning situation or situational problem. It specifies these elements for the situational problem described on the following page.

ELEMENTS NEEDED TO DEVELOP A LEARNING SITUATION OR A SITUATIONAL PROBLEM	
Targeted broad area of learning – Helps contextualize learning and makes it meaningful.	• Career Planning and Entrepreneurship
Prescribed subject-specific competencies – Are developed through the active participation of adult learners.	• Uses strategies to solve situational problems • Uses mathematical reasoning • Communicates by using mathematical language
Prescribed family of learning situations – Consists of real-life situations applicable to a given course. – Helps adult learners acquire mathematical knowledge.	• Relationship between quantities
Targeted cross-curricular competencies – Are developed at the same time and in the same context as the subject-specific competencies.	• Uses information and communications technologies • Uses information
Prescribed essential knowledge – Refers to mathematical knowledge and concepts to be acquired.	• See list

This section provides an example of a situational problem along with possible tasks involved in its mathematical processing. The context can be used as a common thread throughout the learning situation. The learning activities are not spelled out; rather, the focus is on a relevant example of mathematical processing using the four phases in the problem-solving process: representation, planning, activation and reflection. Although not explicitly stated, the elements of the situational problem identified in the previous table, i.e. the broad area of learning, subject-specific competencies, family of learning situations, cross-curricular competencies and essential knowledge, can be discerned, and must form a coherent and meaningful whole.

Teachers may choose to use any of these elements as objects of learning. For instance, learning can focus on actions associated with the phases in the problem-solving process, actions related to the subject-specific or cross-curricular competencies, or actions related to the prescribed knowledge. Teachers can also use the example provided to construct other complex tasks or learning activities related to the mathematical knowledge adult learners must acquire.

Situational problem	Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the ***Relationship between quantities*** family of learning situations
An adult learner wants to find out more about the work of a traffic collision reconstruction expert. She wants to become familiar with the concepts related to this type of reconstruction. In addition to gathering information about a particular event, interpreting physical evidence found at the collision site, taking photographs of the scene and making sketches, such an expert draws on certain mathematical concepts.	**Integrative process:** *Using an algebraic or graphical model of a function to generalize a set of situations* In carrying out the four phases in the problem-solving process, adult learners could: Representation - Select the relevant information (mass and acceleration in this case) and disregard superfluous information (e.g. tire traction, reaction time, type of surface, weather conditions) - Reflect on the need to refer to several similar experiences to be able to come to a generalization Planning - Choose several similar experiments involving acceleration and deceleration - List the elements needed to draw the graph (mass and acceleration in this case)

Situational problem	**Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the *Relationship between quantities* family of learning situations**	
For example, using data resulting from experiments, the adult learner determines the relationship between the acceleration (or deceleration) of a vehicle and its mass and whether it is possible to generalize this rule, especially when the initial speed is changed.	Activation	- Make a table of the data related to the situation, taking into account the limitations and precision of the measuring instruments used - For a given initial speed, find the algebraic rule showing the relationship between acceleration and mass - Repeat the operation with different initial speeds - Compare the resulting relationships in order to derive a general rule of correspondence between acceleration and mass (the rule should be valid regardless of the initial speed)
	Reflection	- Suggest probable or plausible reasons that the equation is not perfectly consistent with the data analyzed (human error, measurement errors, limitations of the instruments used to take the measurements, etc.)

END-OF-COURSE OUTCOMES

To solve situational problems in the family of learning situations *Relationships between quantities*, adult learners represent a situation, interpolate or extrapolate, and generalize a set of situations using an algebraic or graphical model. To do this, they use the three subject-specific competencies, *Uses strategies to solve situational problems*, *Uses mathematical reasoning* and *Communicates by using mathematical language*.

To represent a situational problem using an algebraic or graphical model, adult learners describe, symbolize, code, decode, explain or illustrate the information contained in tables of values or algebraic rules. They combine different registers of representation as needed to produce a message in accordance with the notation, rules and conventions of mathematical language. They use problem-solving strategies to make comparisons, propose corrections, present favourable or optimal solutions, or issue recommendations. They formulate constructive criticism and make informed decisions about issues in a variety of fields, including technical fields (e.g. graphics, biology, physics, administration).

To interpolate or extrapolate results from an algebraic or graphical model in order to make decisions, adult learners use different types of functional and strategic models, combining reasoning and creativity to overcome obstacles. They use structured deductive reasoning and become familiar with the codified form required for their proof. They illustrate, explain or justify their arguments. They use different types of proofs and different lines of reasoning, including proof by exhaustion. The latter is used in particular in analyzing or conducting case studies, or in applying a generalization process leading to the validation of a conjecture. Adult learners also observe specific real-life cases and generalize their observations.

To generalize a set of situations using an algebraic or graphical model, adult learners specify the purpose of their communication and switch from one register to another as needed. They demonstrate their understanding of the problems in question using a wide range of communication strategies, which enables them to regulate their transmission of a message based on the specific reactions of the audience or to take new requirements into account. They learn and correctly use language that appropriately combines everyday, mathematical, technical and scientific terms. They deduce new algebraic rules by combining the different operations on functions they have mastered, and prove them, justifying all the steps in their procedure. In addition, they make effective use of the parameters of the functions to illustrate generalities about a set of functions.

Throughout the problem-solving process, adult learners apply their mathematical knowledge (relations, functions, inverse and system of equations). Their use of symbols, terms and notation related to this knowledge is accurate, and they always refer to different sources to validate the laws, theorems, corollaries or lemmas they deduce or induce so that they can improve their mathematical literacy. In addition, they do not hesitate to ask for help when they encounter difficulties.

EVALUATION CRITERIA FOR THE COMPETENCIES TARGETED BY THE COURSE

Uses strategies to solve situational problems

- *Indication (oral or written) that the situational problem has been understood*
- *Application of strategies and appropriate mathematical knowledge*
- *Development of an appropriate solution**
- *Appropriate validation of the steps** in the solution*

  * The solution includes a procedure, strategies and a final answer.

  ** The mathematical model, operations, properties or relations involved.

Uses mathematical reasoning

- *Formulation of a conjecture suited to the situation*
- *Correct use of appropriate mathematical concepts and processes*
- *Proper implementation of mathematical reasoning suited to the situation*
- *Proper organization of the steps in an appropriate procedure*
- *Correct justification of the steps in an appropriate procedure*

Communicates by using mathematical language

- *Correct interpretation of a mathematical message*
- *Production of a message in keeping with the terminology, rules and conventions of mathematics, and suited to the context*