Course

MTH-4171-2

Algebraic and Graphical Modelling in a Fundamental Context 1

Mathematics

INTRODUCTION

The goal of the *Algebraic and Graphical Modelling in a Fundamental Context 1* course is to enable adult learners to deal with situations that involve using an algebraic or graphical model to represent a dependency relationship between quantities in a fundamental context.

In this course, adult learners expand their knowledge of algebra in order to analyze all the nuances of the functions in question. Step functions give them the opportunity to further develop their understanding of real numbers and their reasoning abilities, especially when they represent and compare *greatest integer, truncation* and *round* functions, as well as the *fractional part* function. In order to determine the parameters of the functions to be studied, adult learners will normally be required to write the standard form of the equation. To make the parameters more meaningful, they analyze their role in the function, their effect on graphs (transformation of the initial function) and their relationship with the given information. Observations and manipulations can be carried out with or without technological tools, depending on the educational goals involved. Technology makes it possible to find the best model more quickly and to focus on its analysis and justification rather than on algebraic manipulations. In situational problems, adult learners identify data (in verbal, algebraic or graphical form or in a table of values), create models, recognize patterns, and interpolate or extrapolate in order to conduct an in-depth analysis of the situation. In many of the situational problems, they must demonstrate their ability to work with algebraic expressions. To arrive at one or more solutions, they use mathematical rigour and deductive reasoning. In addition, the situational problems include tasks in which adult learners must validate and correct the solutions as needed. Other situations involve formal proofs associated with different items of knowledge, in particular the properties and manipulation of algebraic expressions. Still others allow adult learners to analyze a model by determining and interpreting the value of the parameters. Situations involving the concept of correlation help adult learners develop a line of reasoning which, supported by an understanding of dependency relationships and a capacity for abstraction, lead them to recognize cause-and-effect relationships. Situations involving systems of equations or inequalities require that adult learners describe and interpret information. Lastly, some situations require that data be processed in a given register of representation, in particular when writing the rules of second-degree functions in standard, general or factored form, while others require translation from one register to another.

By the end of this course, adult learners will be able to use algebra to represent concrete situations. They will produce clear and accurate work in accordance with the rules and conventions of mathematics. By algebraically or graphically representing a situation using real functions and their inverse, they will be able to employ inductive or deductive reasoning to obtain results through interpolation or extrapolation. In addition, adult learners will use different registers of representation to generalize results and extend them to other situations.

SUBJECT-SPECIFIC COMPETENCIES

In order to solve the situational problems in this course, adult learners will use the following three subject-specific competencies:

- *Uses strategies to solve situational problems*
- *Uses mathematical reasoning*
- *Communicates by using mathematical language*

The use of effective strategies involves employing rigorous mathematical reasoning and communicating clearly by observing the codes and conventions of mathematical language. Adult learners solve situational problems by using all three subject-specific competencies and other resources.

The following section explains how to use the three subject-specific competencies to solve a situational problem.

PROCESS AND STRATEGIES

To solve a situational problem, adult learners need effective strategies that they can adapt to the situations at hand.

Adult learners solve situational problems using a four-phase process:

- **representation**
- **planning**
- **activation**
- **reflection**

The following table gives an overview of the phases in the problem-solving process, as well as a few examples of strategies adult learners can use in dealing with various situations. These phases are not necessarily carried out in the order indicated above. Adult learners may have to go back and forth among the four phases in order to solve a situational problem.

PROCESS AND STRATEGIES	
REPRESENTATION	
- Adult learners examine the situational problem to identify the context, the problem and the task to be performed. - They use observational and representational strategies that are essential to inductive reasoning. - Through different lines of deductive reasoning, they could test their hypothesis by assigning increasingly large values to one variable to determine their effect on the value of the other variable.	
Examples of strategies	• Using a table to display the variables involved • Sketching a Cartesian coordinate graph • Using words or symbols to reformulate the dependency relationship between the variables • Exploring the situational problem by substituting numerical values in algebraic fractions to observe the changes in the quotient
PLANNING	
- In planning their solution, adult learners look for ways of approaching the problem and choose those that seem the most efficient. - Reasoning enables them to establish organized and functional relationships among different aspects of their knowledge, thus expanding their networks of mathematical cognitive resources. - For example, they attempt to extrapolate results using an algebraic rule or a graph.	
Examples of strategies	• Listing the main steps of the solution in point form • Identifying the elements needed for the graphical and algebraic representation of the situation, depending on the register of representation chosen: characteristics of the function, graduation of the axes of the graph, dependent variable and independent variable
ACTIVATION	
- When dealing with a situational problem, adult learners use reasoning to determine the relationships between the change in the parameters of the rule of a function and the transformation of the corresponding Cartesian coordinate graph. - They can also identify the necessary elements such as the scale, properties and constraints related to the domain of the function to draw the graph of the function, bearing in mind the meaning of mathematical symbols, terms and notation.	
Examples of strategies	• Proceeding by trial and error in assigning values to the variables so as to determine the mathematical constraints and properties of algebraic fractions • Solving a problem step by step
REFLECTION	
- Adult learners use a reflective approach throughout the situation and always review the phases in the problem-solving process and the choices made, with a view to validating the solution. - Using reasoning, they make conjectures about particular or special cases in order to validate certain results. - They make sure that the dependent and independent variables are properly defined, that the axes are correctly scaled, that no unit of measure has been omitted and that the data have been correctly transcribed.	
Examples of strategies	• Proceeding by trial and error to determine whether a function is increasing or decreasing by substituting different values into the rule of that function over a given interval • Checking the solution of a system of relations by making sure that an ordered pair in the feasible region is the solution for both the first and second relations • Calling into question the method they have used if negative radicals are obtained in determining the zeros when a real-number solution is expected

CROSS-CURRICULAR COMPETENCIES

Cross-curricular competencies are not developed in a vacuum; they are rooted in situational problems. To varying degrees, the cross-curricular competencies contribute to the development of the subject-specific competencies, and vice versa.

Several cross-curricular competencies can be useful in dealing with the *Relationship between quantities* family of learning situations. Two of these are considered particularly relevant to this course: *Uses information and communications technologies* and *Uses information*.

Methodological Competency

When adult learners are faced with a problem involving any kind of complexity, they need a method of visualizing it to be able to find the appropriate model more quickly. For example, to understand the relationship between the level of radioactivity of uranium and the distance from the mine to their homes, adult learners can use the competency *Uses information and communications technologies* to create and manipulate graphs by modifying some of their parameters. The use of technology allows them to focus on analyzing the situation.

Intellectual Competency

Adult learners are often placed in a learning situation involving an experiment whose results have already been published. When searching for information related to the situation they wish to model, they discover a large amount of data that are contradictory or of unequal value. This is an ideal opportunity to apply the competency *Uses information*. In looking for appropriate information, they assess the reliability of their sources. In addition, they must organize this information, a process that calls for a great deal of intellectual rigour.

SUBJECT-SPECIFIC CONTENT

In this course, adult learners use and build on their previously acquired knowledge of arithmetic and algebra. In order to deal effectively with situational problems, they will add to what they have learned by mastering the mathematical knowledge specific to this course.

Prescribed Knowledge

In order to deal effectively with the learning situations in this course, adult learners develop the following three integrative processes:

- **using an algebraic or graphical model to represent a situation**
- **interpolating or extrapolating from an algebraic or graphical model**
- **using an algebraic or graphical model to generalize a set of situations**

These processes, which are applied in the learning situations in this course, foster the integration of mathematical knowledge and the subject-specific competencies. The learning situations must involve at least one of these integrative processes. However, there must be a sufficient variety of learning situations to cover all three processes.

Mathematical Knowledge	Restrictions and Clarifications
Manipulating algebraic expressions	
• Operations on algebraic expressions	Operations on algebraic expressions are limited to: • multiplication • division of polynomials by a binomial (with or without a remainder) • the simplification of rational expressions (rational fractions)
• Expanding, simplifying or substituting expressions using significant algebraic identities	The significant algebraic identities of the second degree are: • perfect square trinomial • difference of two squares
• Completing the square	Completing the square is used for factoring second-degree polynomial functions and switching from one way of writing them to another.
• Factoring trinomials using roots	Trinomials are factored using the roots of the polynomial, if any: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

<table>
<tr><th>Mathematical Knowledge</th><th>Restrictions and Clarifications</th></tr>
<tr><td>Manipulating algebraic expressions (cont.)
Solving first-degree equations and inequalities in one or two variables and second-degree equations and inequalities in one variable</td><td>Equations and inequalities are solved:
algebraically
graphically</td></tr>
<tr><td>Function
Experimenting with real functions as well as observing, interpreting, describing and representing them</td><td>The real functions studied in this course are:
second-degree polynomials
general form $f(x) = ax^2 + bx + c$
factored form $f(x) = a(x - x_1)(x - x_2)$
standard form $f(x) = a(x - h)^2 + k$
step functions (greatest integer not greater than x)
$f(x) = a[b(x - h)] + k$
Functions can be represented:
verbally
using a table of values
algebraically
graphically, with or without the use of technology</td></tr>
<tr><td>Describing and interpreting the properties of real functions</td><td>The properties of real functions covered in this course are:
domain and codomain (range)
increasing and decreasing intervals
extrema
sign
x- and y-intercepts</td></tr>
<tr><td>Interpreting the multiplicative and additive parameters
Switching from one form to another in writing second-degree polynomial functions</td><td></td></tr>
</table>

Mathematical Knowledge	Restrictions and Clarifications
System	
• Representing a situation using straight lines or half-planes	The properties of the following lines are studied: • parallel lines • intersecting lines • coincident lines • perpendicular lines The equation of the line can be expressed in: • general form $Ax + By + C = 0$ • standard form $y = ax + b$ • symmetric form $\left(\frac{x}{a} + \frac{y}{b}\right) = 1$
• Solving systems of first-degree equations in two variables • Solving systems composed of a first-degree equation and a second-degree equation in two variables	Systems may be solved: ▪ using a table of values ▪ algebraically ▪ graphically

Cultural References

Mathematics is responsible for the existence and effectiveness of a growing number of objects, tools and techniques used on a daily basis. For example, recent developments in weather forecasting, digital image processing, data fusion related to aerial and space-based surveillance, the control of rail transport, the optimization of cellular telephone networks and hydroelectric management of a power station or region all involve mathematical modelling.

Mathematical modelling also makes it possible to understand why a hydroelectric dam shaped like a parabola can better withstand the enormous pressure exerted by the water in the upstream reservoir than a dam of another shape. Graphical representation can enable adult learners to gain insight into the importance of mathematics and modelling in building structures. They could also compare present-day structures with those built a thousand years ago and discover that these principles were already known back then.

Adult learners can explore numerous examples of phenomena modelled using algebraic equations and realize that algebraic concepts are used in many fields such as air traffic control, operational research, computer science, cryptography and economics, to name just a few.

FAMILY OF LEARNING SITUATIONS

The situations in the family *Relationship between quantities* involve problems that can be solved in part by using an algebraic or graphical model to represent a relationship between quantities in a fundamental context. The *Algebraic and Graphical Modelling in a Fundamental Context 1* course provides adult learners with an opportunity to express a connection or a dependency relationship between quantities.

In the situational problems in this course, adult learners test their hypothesis by assigning increasingly large values to one variable to determine their effect on the value of the other variable, determine the relationships between the change in the parameters of the rule of a function and the transformation of the corresponding Cartesian coordinate graph, and distinguish between the mathematical and everyday meanings of the terms used.

BROAD AREAS OF LEARNING

The broad areas of learning deal with major contemporary issues. Ideally, the situations to be studied should be selected in keeping with the educational aims of the broad areas of learning, which provide the situational problems with contexts that make the learning process meaningful. Two broad areas of learning are considered particularly relevant to this course: Media Literacy, and Environmental Awareness and Consumer Rights and Responsibilities.

Media Literacy

Adult learners interested in car chases could conduct a study on the variation in parameters such as maximum speed and acceleration in order to derive algebraic equations from graphical models. They could then prepare a multimedia presentation to explain the conclusions they have drawn from their observations. By acting ethically, adult learners can help their audience distinguish between facts and misconceptions, thereby fostering an understanding of media representations of reality. This is one of the focuses of development of this broad area of learning.

Environmental Awareness and Consumer Rights and Responsibilities

Adult learners who are concerned about environmental problems and interested in systems for generating renewable energy could use certain functions to analyze the efficiency of a wind turbine or a solar panel in relation to their cost. The performance of photovoltaic cells varies with the surface area of the panel, the amount of incident sunlight and various other factors. Adult learners could estimate their electricity requirements and, using graphs or tables of values, explore the potential merit of investing in the technology or select the size of the solar panel or wind turbine that will meet their needs. This could make them aware of their electricity consumption and prompt them to consider alternatives for using resources wisely. In this way, they could make informed choices, which ties in with one of the focuses of development of this broad area of learning.

EXAMPLE OF A LEARNING SITUATION

All learning situations and situational problems, regardless of the broad area of learning to which they are related, require the active participation of the adult learner. They provide an opportunity to develop the targeted subject-specific and cross-curricular competencies, to acquire mathematical concepts and to mobilize a variety of useful resources.

The table below presents the elements needed to develop a learning situation or situational problem. It specifies these elements for the situational problem described on the following page.

ELEMENTS NEEDED TO DEVELOP A LEARNING SITUATION OR A SITUATIONAL PROBLEM	
Targeted broad area of learning – Helps contextualize learning and makes it meaningful.	• Media Literacy
Prescribed subject-specific competencies – Are developed through the active participation of adult learners.	• Uses strategies to solve situational problems • Uses mathematical reasoning • Communicates by using mathematical language
Prescribed family of learning situations – Consists of real-life situations applicable to a given course. – Helps adult learners acquire mathematical knowledge.	• Relationship between quantities
Targeted cross-curricular competencies – Are developed at the same time and in the same context as the subject-specific competencies.	• Uses information and communications technologies • Uses information
Prescribed essential knowledge – Refers to mathematical knowledge and concepts to be acquired.	• See list

This section provides an example of a situational problem along with possible tasks involved in its mathematical processing. The context can be used as a common thread throughout the learning situation. The learning activities are not spelled out; rather, the focus is on a relevant example of mathematical processing using the four phases in the problem-solving process: representation, planning, activation and reflection. Although not explicitly stated, the elements of the situational problem identified in the previous table, i.e. the broad area of learning, subject-specific competencies, family of learning situations, cross-curricular competencies and essential knowledge, can be discerned, and must form a coherent and meaningful whole.

Teachers may choose to use any of these elements as objects of learning. For instance, learning can focus on actions associated with the phases in the problem-solving process, actions related to the subject-specific or cross-curricular competencies, or actions related to the prescribed knowledge. Teachers can also use the example provided to construct other complex tasks or learning activities related to the mathematical knowledge adult learners must acquire.

<table>
<tr><th>Situational problem</th><th colspan="2">Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the Relationship between quantities family of learning situations</th></tr>
<tr><td rowspan="3">Car chases are very popular in movies. One car is travelling along a highway and, far behind it, another car sets off in pursuit. When the driver of the first car realizes he is being pursued, he accelerates in an effort to get away. How will the chase unfold?</td><td colspan="2">Integrative process: Using an algebraic or graphical model to generalize a set of situations
In carrying out the four phases in the problem-solving process, adult learners could:</td></tr>
<tr><td>Representation</td><td>- Describe the characteristics of the situation in their own words to have a clear idea of the problem
- Put forward a hypothesis about the relationship between the position of the cars and time
- Make an intuitive assumption that if the acceleration increases, the minimum distance between the two cars will decrease</td></tr>
<tr><td>Planning</td><td>- Refer to a similar situational problem already analyzed in class when modelling the relations on the basis of the data provided
- Determine the order in which the different steps are carried out: for example, plot the given data on a graph before looking for the relationship between the data and then generalize the system of equations</td></tr>
</table>

<table>
<tr><th>Situational problem</th><th colspan="2">Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the Relationship between quantities family of learning situations</th></tr>
<tr><td rowspan="2">It is assumed that the two cars have the same maximum speed and that the car being pursued has a constant acceleration until it reaches its maximum speed. Adult learners are provided with enough data on the two cars to determine the type of relationship that exists between the position of the cars and their travel time.

Adult learners are asked to generalize this type of situation by establishing a system of equations and to determine what the initial minimum distance between the two cars would have to be to enable the first car to accelerate and get away. The speed of the car in pursuit is assumed to be constant.

Adult learners will have to give a multimedia presentation to show their findings.</td><td>Activation</td><td>- Sketch a Cartesian coordinate graph showing the position of the two cars as a function of time
- Determine the algebraic rule for each car
- Determine through extrapolation when the two vehicles will meet
- Determine what the initial distance would have to be in order for the first car to get away (using technology if possible)
- Vary the acceleration and modify the graph accordingly, and then determine the minimum initial distance between the two cars that would enable the first car to get away (as a function of the speed of the pursuer and the acceleration of the car being pursued)
- Repeat this final step with a view to generalizing the system of equations
- Use formal mathematical language to generalize the situation</td></tr>
<tr><td>Reflection</td><td>- Compare their solution and results with those of their classmates in order to identify the strengths and weaknesses of their model
- Examine whether or not they have made appropriate choices: could the second-degree function have been replaced by an exponential function and, if so, why? Is the initial assumption (the minimum distance decreases if acceleration increases) correct?
- Answer the following question: would the modification of a parameter other than the initial distance between the cars make it possible for the car being pursued to get away?</td></tr>
</table>

END-OF-COURSE OUTCOMES

To solve situational problems in the family of learning situations *Relationships between quantities*, adult learners represent a situation, interpolate or extrapolate, and generalize a set of situations using an algebraic or graphical model. To do this, they use the three subject-specific competencies, *Uses strategies to solve situational problems*, *Uses mathematical reasoning* and *Communicates by using mathematical language*.

To represent a situational problem using an algebraic or graphical model, adult learners use real functions and their inverse, and select relevant information in order to derive a pattern or law that will take into account the best relationship between the constraints and consequences of the situation. They choose the algebraic model best suited to the situation, giving examples with numerical values as needed in order to make a decision concerning the type of relation that exists between the variables of the situation. In addition, they recognize and choose mathematical symbols, terms and notation with a view to representing a situation accurately. They produce accurate mathematical messages in accordance with the mathematical rules and conventions associated with the functions covered in the course. If they are required to solve systems of first- and second-degree equations, they validate their solutions or intuitions algebraically, sometimes using a graph. They are also able to justify all of the steps in their process using mathematical language.

Adult learners formulate questions related to the situation by interpolating or extrapolating from an algebraic or graphical model. These questions help them establish organized and functional relationships among different aspects of mathematical knowledge (e.g. the relationships between the parameters of a given function or how a change in a parameter affects a family of functions). Adult learners suggest probable or plausible ideas in order to deduce propositions related to the situation. They then validate their conjectures through interpolation or extrapolation, substituting numerical values in the algebraic rule they have modelled. They find the rule using the zeros of the function or the characteristics of the step function.

When modelling several situations using a real function, adult learners consider whether or not their properties can be generalized. To do this, they determine the important elements and the obstacles to be overcome and refer to the solution of one or more similar situational problems. They find invariants through trial and error, which enables them to make generalizations that lead to laws, rules or properties. They validate their solution using examples or counterexamples to test their deductive reasoning. In addition, solving systems of first-degree equations in two variables provides the tools they need to generalize results that lead to the properties of different types of lines, be they parallel, perpendicular, coincident or intersecting. Lastly, to perform operations on algebraic expressions, adult learners use factoring involving significant identities: perfect square trinomial or difference of two squares. They easily identify the specific characteristics of algebraic fractions, and can illustrate their conclusions using a graph.

Throughout the problem-solving process, adult learners apply their mathematical knowledge (manipulation of numerical and algebraic expressions, function, inverse and system). Their use of symbols, terms and notation related to this knowledge is accurate, and they always refer to different sources to validate the laws, theorems, corollaries or lemmas they deduce or induce so that they can improve their mathematical literacy. In addition, they do not hesitate to ask for help when they encounter difficulties.

EVALUATION CRITERIA FOR THE COMPETENCIES TARGETED BY THE COURSE

Uses strategies to solve situational problems

- *Indication (oral or written) that the situational problem has been understood*
- *Application of strategies and appropriate mathematical knowledge*
- *Development of an appropriate solution**
- *Appropriate validation of the steps** in the solution*

  * The solution includes a procedure, strategies and a final answer.

  ** The mathematical model, operations, properties or relations involved.

Uses mathematical reasoning

- *Formulation of a conjecture suited to the situation*
- *Correct use of appropriate mathematical concepts and processes*
- *Proper implementation of mathematical reasoning suited to the situation*
- *Proper organization of the steps in an appropriate procedure*
- *Correct justification of the steps in an appropriate procedure*

Communicates by using mathematical language

- *Correct interpretation of a mathematical message*
- *Production of a message in keeping with the terminology, rules and conventions of mathematics, and suited to the context*