Course

MTH-4161-2

Algebraic and Graphical Modelling in an Applied Context 1

Mathematics

INTRODUCTION

The goal of the *Algebraic and Graphical Modelling in an Applied Context 1* course is to enable adult learners to deal with situations that involve using an algebraic or graphical model to represent a dependency relationship between quantities in an applied context.

In this course, adult learners solve situational problems in which they must ask questions when it comes to choosing the most appropriate function. For example, does one use a step function, a first-degree function or a piecewise function to represent a per-minute long-distance telephone rate? Adults learn to analyze concrete situations using periodic functions, piecewise functions or step functions. While square root and logarithmic functions are represented graphically, the associated concepts are usually taught as inverse operations involved in solving second-degree or exponential equations and inequalities related to a given situation. Operations on functions can be approached intuitively, as needed. In some situations, adult learners produce, analyze or compare parts of a bid that require mathematical processing. They can use their critical judgment to analyze drawings, algorithms or possible solutions in order to evaluate efficiency and, if applicable, to identify errors and anomalies, take corrective measures, suggest improvements or issue recommendations. Other situations, when necessary, involve using the appropriate instrument to work out a solution, taking into account the precision with which the solution can be validated. In some situational problems, adult learners perform a variety of mental operations involving comparison, exploration, experimentation and simulation. These operations enable them to make conjectures, come up with an interpretation or conclusion, or establish proofs. Many contexts require a thorough knowledge of the concepts and processes needed to use reasoning to compare and comment on solutions, identify errors and anomalies, and suggest changes in line with the objectives of the situation. The situations proposed require that adult learners produce structured explanations or justifications in order to illustrate how they reached their conclusions. With respect to communicating using mathematical language, some situational problems require that adult learners identify and analyze how someone else’s reasoning is organized. In the situational problems in this course, adult learners share information, a description, an explanation or an argument verbally or in writing, by developing either an activity, a communication plan or a report on an experimental procedure (e.g. laboratory report, logbook).

By the end of this course, adult learners will be able to use algebra to represent concrete situations and produce clear and accurate work in accordance with the rules and conventions of mathematics. By algebraically or graphically representing a situation using real functions or their inverse, they will be able to deduce results through interpolation or extrapolation. In addition, they will use different registers of representation to generalize the similar characteristics of a range of situations.

SUBJECT-SPECIFIC COMPETENCIES

In order to solve the situational problems in this course, adult learners will use the following three subject-specific competencies:

- *Uses strategies to solve situational problems*
- *Uses mathematical reasoning*
- *Communicates by using mathematical language*

The use of effective strategies involves employing rigorous mathematical reasoning and communicating clearly by observing the codes and conventions of mathematical language. Adult learners solve situational problems by using all three subject-specific competencies and other resources.

The following section explains how to use the three subject-specific competencies to solve a situational problem.

PROCESS AND STRATEGIES

To solve a situational problem, adult learners need effective strategies that they can adapt to the situations at hand.

Adult learners solve situational problems using a four-phase process:

- **representation**
- **planning**
- **activation**
- **reflection**

The following table gives an overview of the phases in the problem-solving process, as well as a few examples of strategies adult learners can use in dealing with various situations. These phases are not necessarily carried out in the order indicated above. Adult learners may have to go back and forth among the four phases in order to solve a situational problem.

<table>
<tr><th colspan="2">PROCESS AND STRATEGIES</th></tr>
<tr><th colspan="2">REPRESENTATION</th></tr>
<tr><td colspan="2">- Adult learners examine the situational problem to identify the context, the problem and the task to be performed.
- They use observational and representational strategies that are essential to inductive reasoning.
- Using representational strategies, they see whether there are any trends or patterns and determine if they persist for each iteration. Different lines of deductive reasoning could lead them to generalizations.</td></tr>
<tr><td>Examples of strategies</td><td>• Writing literal expressions to represent the elements of the situation that seem relevant, thus making it easier to find a dependency relationship and determine the variables in the situation
• Using examples involving numbers, determining the type of relationship that exists between the variables in a situation
• Gathering relevant information</td></tr>
<tr><th colspan="2">PLANNING</th></tr>
<tr><td colspan="2">- In planning their solution, adult learners look for ways of approaching the problem and choose those that seem the most efficient.
- They use different types of reasoning to work out the steps involved. They can refer to similar situations they have resolved in the past.
- In order to better understand the relationship between the quantities, they could use a table or a graph to represent a relation.</td></tr>
<tr><td>Examples of strategies</td><td>• Systematically determining the functional model best suited to the situation, bearing in mind the limitations regarding the model's precision
• Finding an algebraic rule that reflects the best relationship between the constraints and possible consequences of the situational problem</td></tr>
<tr><th colspan="2">ACTIVATION</th></tr>
<tr><td colspan="2">- When dealing with a situational problem, adult learners use reasoning to establish organized and functional relationships among different aspects of their knowledge, thus expanding their networks of mathematical cognitive resources.
- They use the appropriate scale so that the graph they draw in solving the situational problem makes sense in light of the context.</td></tr>
<tr><td>Examples of strategies</td><td>• Proceeding by trial and error to determine certain properties of functions
• Dividing the situational problem into subproblems to work out a solution
• Deducing the positive interval of the function by progression</td></tr>
<tr><th colspan="2">REFLECTION</th></tr>
<tr><td colspan="2">- Adult learners use a reflective approach throughout the situation and always review the phases in the problem-solving process and the choices made, with a view to validating the solution.
- They make sure that the dependent and independent variables are correctly defined, that the axes are properly scaled, that no unit of measure has been omitted and that the data have been accurately transcribed.</td></tr>
<tr><td>Examples of strategies</td><td>• Checking their solution by, for example, making sure that the resulting values satisfy the range of the function, comparing the number of possible solutions for a system of equations with the number of solutions found, or substituting the values of the variables in the algebraic expression in order to validate a graphical interpolation or extrapolation.</td></tr>
</table>

CROSS-CURRICULAR COMPETENCIES

Cross-curricular competencies are not developed in a vacuum; they are rooted in situational problems. To varying degrees, the cross-curricular competencies contribute to the development of the subject-specific competencies, and vice versa.

Several cross-curricular competencies can be useful in dealing with the family of learning situations *Relationship between quantities*. Two of these are considered particularly relevant to this course: *Uses information and communications technologies* and *Adopts effective work methods*.

Methodological Competencies

Calculation software such as a spreadsheet program can be used to develop a graphical model to represent a situation. This tool can make it considerably easier to design and carry out tasks, providing more time for the analysis and interpretation of changes in certain parameters of functions. The development of the competency *Uses information and communications technologies* could help adult learners use this tool to work with parameters.

When studying functions and generalization by means of a functional model, especially when this model is constructed using experimental data, adult learners must work systematically in planning tasks and taking measurements. The development of the competency *Adopts effective work methods* is particularly important for adult learners who wish to study in scientific or technical fields.

SUBJECT-SPECIFIC CONTENT

In this course, adult learners use and build on their previously acquired knowledge of arithmetic and algebra. In order to deal effectively with situational problems, they will add to what they have learned by mastering the mathematical knowledge specific to this course.

Prescribed Knowledge

In order to deal effectively with the learning situations in this course, adult learners develop the following three integrative processes:

- **using an algebraic or graphical model to represent a situation**
- **interpolating or extrapolating from an algebraic or graphical model**
- **using an algebraic or graphical model to generalize a set of situations**

These processes, which are applied in the learning situations in this course, foster the integration of mathematical knowledge and the subject-specific competencies. The learning situations must involve at least one of these integrative processes. However, there must be a sufficient variety of learning situations to cover all three processes.

Mathematical Knowledge	Restrictions and Clarifications
Manipulating numerical and algebraic expressions	
• Operations on numerical and algebraic expressions	Operations on algebraic expressions are limited to: • multiplication • division of polynomials by a binomial (with or without a remainder) • the simplification of rational expressions (rational fractions) *When finding the common denominator in order to add two rational expressions, adult learners in this option will deal only with the case in which the denominator of one fraction is the multiple of the denominator of the other fraction.* The numbers can be expressed using: • rational exponents • radicals (n^{th} root) • powers of bases 2 and 10 (base change)
• Constructing and interpreting tables of values consisting of positive rational numbers written in base 2 and base 10	In finding the approximate value of an exponent (logarithm), adult learners use a graph, a table of values (base 2 or 10) or technology. They manipulate expressions and convert them to the same base (base 10, for the calculator) so that the exponents can be compared. If necessary, they use equivalences such as: • $a^b = c \Leftrightarrow \log_a c = b$ • $\log_a c = \frac{\log c}{\log a}$
• Expanding and factoring	The types of factoring covered in this course are factoring by grouping and the use of second-degree algebraic identities (perfect square trinomial and difference of two squares).

<table>
<tr><th>Mathematical Knowledge</th><th>Restrictions and Clarifications</th></tr>
<tr><td>Manipulating numerical and algebraic expressions (cont.)
Solving equations and inequalities in one variable: second-degree, square root, exponential, logarithmic (including the properties of radicals, exponents and logarithms)</td><td></td></tr>
<tr><td>Function and inverse
Experimenting with real functions as well as observing, interpreting, describing and representing them</td><td>The real functions covered are:
second-degree polynomial functions
$f(x) = ax^2$ or $f(x) = (bx)^2$ or $f(x) = a(bx)^2$
exponential functions
$f(x) = ac^{bx}$ where $a \neq 0$ and $c > 0$
square-root functions
$f(x) = a\sqrt{bx}$
This function is introduced in connection with second-degree functions (as an inverse).
periodic functions
logarithmic functions
$f(x) = a\log_c bx$ where $c > 0$
This function is introduced in connection with exponential functions (as an inverse).
greatest-integer functions
$f(x) = a[bx]$
piecewise functions
step functions
The functions can be represented using:
a table of values
an algebraic rule
a graph (produced with or without technology)</td></tr>
</table>

Mathematical Knowledge	Restrictions and Clarifications
Function and inverse (cont.) • Solving and graphing first-degree inequalities in two variables	
• Describing and interpreting the properties of real functions	The properties of real functions studied in this course are: • the domain and codomain (range) • increasing and decreasing intervals • extrema • sign • x- and y-intercepts
• Interpreting the multiplicative parameter	
System • Representing a situation using straight lines or half-planes	The course covers the properties of the following lines: • parallel lines • intersecting lines • coincident lines • perpendicular lines The equation of a line can be expressed in: • general form $Ax + By + C = 0$ • standard form $f(x) = ax + b$ *The symmetric form* $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$ *of the equation of a line is optional in the* Technical and Scientific *option.*
• Solving systems of first-degree equations in two variables	System of equations may be solved by means of: • a table of values • an algebraic method chosen by the adult learner • a graphical method, with or without the use of technology

Cultural References

Modelling is a way of understanding reality and establishing dependency relationships between quantities. Dependency relationships led to the development of the concepts of relation and function. The history of mathematics shows that Diophantus of Alexandria defined the concept of an unknown number more than 18 centuries ago. Diophantus even went so far as to work with up to ten unknowns. Mathematical concepts have sometimes given rise to a great deal of controversy before gaining acceptance, and their development has resulted from intellectual jousting between philosophers and scientists throughout history. When a concept is finally attributed to a given mathematician, it is often the case that many other people worked on it beforehand. For example, many mathematicians contributed to the development of algebra, including Oresme, who established the equation of a line three centuries before Descartes invented analytic geometry.

Mathematical reasoning and geometric modelling are used to design many commonly used instruments, and their use requires graphical representations. The evolution of this branch of mathematics has played a vital role in the development of several of today's essential instruments (e.g. sphygmomanometer used to measure blood pressure, radar, multimeter). Algebraic and graphical modelling is of prime importance when using these instruments in professional or technical occupations in the sciences. Examples given during the course will enable adult learners to understand their importance.

Analytic geometry has made it possible to represent many physical phenomena. Depending on their interests, adult learners could use this mathematical approach in a project to study the behaviour of a golf ball, the orbit of the planets in our solar system or the operation of a measuring instrument. They could then use equations to represent these physical phenomena and analyze them to derive useful information.

FAMILY OF LEARNING SITUATIONS

The situations in the family *Relationship between quantities* involve problems that can be solved in part by using an algebraic or graphical model to represent a relationship between quantities in an applied context. The *Algebraic and Graphical Modelling in an Applied Context 1* course provides adult learners with an opportunity to express a connection or a dependency relationship between quantities.

In the situational problems in this course, adult learners see whether there are any trends or patterns and determine if they persist for each iteration, derive and generalize the rules and conditions that determine the number of solutions for the system, and make sure that the dependent and independent variables are correctly defined, that the axes are properly scaled, that no unit of measure has been omitted, and that the data have been accurately transcribed.

BROAD AREAS OF LEARNING

The broad areas of learning deal with major contemporary issues. Ideally, the situations to be studied should be selected in keeping with the educational aims of the broad areas of learning, which provide the situational problems with contexts that make the learning process meaningful. Two broad areas of learning are considered particularly relevant to this course: Health and Well-Being, and Environmental Awareness and Consumer Rights and Responsibilities.

Health and Well-Being

This course could help adult learners understand the way bacteria spread on door handles, telephones or computer mice and keyboards. After collecting data during an experiment, they could graph the increase in the number of bacteria as a function of time or the number of users. They could also compile their results in a table of values and graph them. Their conclusions could provide an explanation for the transmission of bacteria in different environments and encourage them to extrapolate in order to simulate an epidemic and become aware of the importance of a healthy lifestyle, which ties in with the educational aim of this broad area of learning.

Environmental Awareness and Consumer Rights and Responsibilities

The study of functions could also help adult learners understand the consequences of their driving habits. For example, by analyzing their gas consumption as a function of their driving speed, they would be able to assess the cost and energy expenditure that result from driving fast. Moreover, they could calculate braking distance as a function of speed. In short, adult learners could be encouraged to take a critical attitude toward consumption and the exploitation of the environment, which ties in with the educational aim of this broad area of learning.

EXAMPLE OF A LEARNING SITUATION

All learning situations and situational problems, regardless of the broad area of learning to which they are related, require the active participation of the adult learner. They provide an opportunity to develop the targeted subject-specific and cross-curricular competencies, to acquire mathematical concepts and to mobilize a variety of useful resources.

The table below presents the elements needed to develop a learning situation or situational problem. It specifies these elements for the situational problem described on the following page.

ELEMENTS NEEDED TO DEVELOP A LEARNING SITUATION OR A SITUATIONAL PROBLEM	
Targeted broad area of learning – Helps contextualize learning and makes it meaningful.	• Health and Well-Being
Prescribed subject-specific competencies – Are developed through the active participation of adult learners.	• Uses strategies to solve situational problems • Uses mathematical reasoning • Communicates by using mathematical language
Prescribed family of learning situations – Consists of real-life situations applicable to a given course. – Helps adult learners acquire mathematical knowledge.	• Relationship between quantities
Targeted cross-curricular competencies – Are developed at the same time and in the same context as the subject-specific competencies.	• Uses information and communications technologies • Adopts effective work methods
Prescribed essential knowledge – Refers to mathematical knowledge and concepts to be acquired.	• See list

This section provides an example of a situational problem along with possible tasks involved in its mathematical processing. The context can be used as a common thread throughout the learning situation. The learning activities are not spelled out; rather, the focus is on a relevant example of mathematical processing using the four phases in the problem-solving process: representation, planning, activation and reflection. Although not explicitly stated, the elements of the situational problem identified in the previous table, i.e. the broad area of learning, subject-specific competencies, family of learning situations, cross-curricular competencies and essential knowledge, can be discerned, and must form a coherent and meaningful whole.

Teachers may choose to use any of these elements as objects of learning. For instance, learning can focus on actions associated with the phases in the problem-solving process, actions related to the subject-specific or cross-curricular competencies, or actions related to the prescribed knowledge. Teachers can also use the example provided to construct other complex tasks or learning activities related to the mathematical knowledge adult learners must acquire.

Situational problem	**Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the *Relationship between quantities* family of learning situations**
In 2006, a study conducted in Québec showed that the *C. difficile* bacterium, the main cause of infectious diarrhea among hospital patients in industrialized countries, was indirectly responsible for 108 deaths in six months. This infection, which is the most common in hospitals and long-term care facilities, also affects hospital staff. As is the case for all infectious diseases, one of the best means of preventing infection is to frequently wash one's hands with warm soapy water for at least 20 seconds.	**Integrative process:** *Using an algebraic or graphical model to represent a situation* In carrying out the four phases in the problem-solving process, adult learners could: Representation - Describe the situation in their own words - Put forward a hypothesis on the type of relationship that exists between the number of bacteria and elapsed time (if the number of people who touch the door handle doubles, the number of bacteria will also double) and then confirm or refute their hypothesis Planning - Refer to a similar situational problem previously studied in class (pyramid sales schemes, uncontrolled chain reaction resulting from nuclear fission) to begin their analysis - Decide, according to the information provided, to begin by representing the relationship using a graphical model rather than an algebraic one

Situational problem		Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the *Relationship between quantities* family of learning situations
Adult learners are asked to become aware of the importance of being careful about occupational health and safety and to raise awareness of this issue. They will have to show the importance of hand-washing by illustrating the phenomenal growth of harmful bacteria on door handles. For their demonstration, adult learners will use data provided by their teacher or found on the Internet.	Activation	- Determine which is the independent (number of bacteria) and the dependent (time elapsed) variable - Draw a Cartesian coordinate graph of the situation, using an appropriate scale - Draw a curve through the data values - Associate this curve with an exponential relationship
	Reflection	- Compare their solution and results with those of their teacher or classmates in order to identify the strengths and weaknesses of their model - Determine whether there is a critical number beyond which the situation no longer reflects the algebraic model - Check whether the bacterial growth model is still valid under different conditions (e.g. if the temperature increases or decreases markedly)

END-OF-COURSE OUTCOMES

To solve situational problems in the family of learning situations *Relationship between quantities*, adult learners represent a situation, interpolate or extrapolate, and generalize a set of situations using an algebraic or graphical model. To do this, they use the three subject-specific competencies, *Uses strategies to solve situational problems*, *Uses mathematical reasoning* and *Communicates by using mathematical language*.

To represent a situational problem using an algebraic or graphical model, adult learners employ real functions or their inverse, and select relevant information in order to find a pattern or a law that takes into account the best relationship between the constraints and consequences involved. They choose the algebraic model that is best suited to the situation, giving examples with numerical values as needed in order to determine the type of relationship that exists between the variables. In addition, they recognize and choose the mathematical symbols, terms and notation with a view to representing the situation accurately. They produce precise mathematical messages using the rules and conventions specific to the functions covered in this course.

Adult learners determine questions before interpolating or extrapolating from an algebraic or graphical model. These questions help them establish organized and functional relationships among certain aspects of mathematical knowledge, in particular the relationships between the parameters of a given function or the effect of a change in a parameter on a family of functions. They then suggest probable or plausible ideas in order to deduce propositions related to the situation and validate their conjectures through interpolation or extrapolation, substituting numerical values in the algebraic rule they have modelled. They also associate the appropriate graph with the sequences of numbers in question (unlike a geometric progression, which must be graphed using an exponential relation, an arithmetic progression can be graphed using a linear relation).

Adult learners model several situations using real functions, and consider whether or not the properties of these situations can be generalized. To do this, they must determine the important elements and the obstacles to be overcome and refer to the solution of one or more similar situational problems. They find invariants through trial and error, which enables them to make generalizations and derive laws, rules or properties. They validate their solution using examples or counterexamples to test their deductive reasoning. In addition, solving systems of first-degree equations in two variables provides them with the tools they need to generalize results that lead to the properties of different types of lines, be they parallel, perpendicular, coincident or intersecting.

Throughout the problem-solving process, adult learners apply their mathematical knowledge (manipulating numerical and algebraic expressions, function, inverse and system). Their use of symbols, terms and notation related to this knowledge is accurate, and they always refer to different sources to validate the laws, theorems, corollaries or lemmas they deduce or induce so that they can improve their mathematical literacy. In addition, they do not hesitate to ask for help when they encounter difficulties.

EVALUATION CRITERIA FOR THE COMPETENCIES TARGETED BY THE COURSE

Uses strategies to solve situational problems

- *Indication (oral or written) that the situational problem has been understood*
- *Application of strategies and appropriate mathematical knowledge*
- *Development of an appropriate solution**
- *Appropriate validation of the steps** in the solution*

  * The solution includes a procedure, strategies and a final answer.

  ** The mathematical model, operations, properties or relations involved.

Uses mathematical reasoning

- *Formulation of a conjecture suited to the situation*
- *Correct use of appropriate mathematical concepts and processes*
- *Proper implementation of mathematical reasoning suited to the situation*
- *Proper organization of the steps in an appropriate procedure*
- *Correct justification of the steps in an appropriate procedure*

Communicates by using mathematical language

- *Correct interpretation of a mathematical message*
- *Production of a message in keeping with the terminology, rules and conventions of mathematics, and suited to the context*