Course

MTH-5151-1

Algebraic and Graphical Modelling in a General Context 2

Mathematics

MTH-5151-1 Algebraic and Graphical Modelling in a General Context 2

INTRODUCTION

The goal of the *Algebraic and Graphical Modelling in a General Context 2* course is to enable adult learners to deal with situations that involve using an algebraic or graphical model to represent a dependency relationship between quantities in a general context.

In Secondary V, adult learners continue to study real numbers and to expand their knowledge of powers and logarithms. Some of the situations they encounter may require them to determine the approximate value of an exponent (logarithm) by using a graph, a table of values or a calculator. They may have to change exponential notation to logarithmic notation and vice versa.

In some situations, adult learners apply their knowledge of real functions such as the second-degree polynomial function ($f(x) = ax^2$, where $a \neq 0$) and the exponential function ($f(x) = ab^x$, where $a \neq 0$ and $b > 0$). The purpose of these functions is to help adult learners to compare, analyze and recognize the characteristics of the curve in order to be able to select the function that is best suited to the situation. Thus, adult learners may use such strategies as the second difference of the *y*-values (second-degree polynomial function) or strategies that involve multiplication of *y*-values when the data are presented in a table of values (exponential function). With respect to the real functions previously seen in the course *Algebraic and Graphical Modelling in a General Context 1* (MTH-4151-1), adult learners may be required to calculate values, make graphs and analyze the properties of a particular function, without having to provide an algebraic solution to the situational problem.

Furthermore, the purpose of this course is to introduce adult learners to financial mathematics and to familiarize them with the related vocabulary. Adult learners will be required to calculate and analyze the future value of a sum of money (capital) invested for one period at a fixed annual interest rate as well as determine the future value of capital invested at an annual compound interest rate for several periods. Adult learners may also be required to compare interest rates with a view to determining the most advantageous one so that they can make informed decisions.

At the end of this course, adult learners will be able to represent concrete situations using exponents or logarithms and to analyze economic (e.g. personal finances), social, technical and everyday situations. They will produce clear and accurate work in accordance with the rules and conventions of mathematics. By algebraically or graphically representing a situation using real functions and operations on these functions, they will be able to induce results through interpolation or extrapolation. They can interpolate or extrapolate using a table of values, a graph or algebra when the algebraic rule is given. Lastly, adult learners will use different registers of representation (tables of values, graphs or algebraic rules) to generalize a model so that it can be applied to a range of situations.

SUBJECT-SPECIFIC COMPETENCIES

In order to solve the situational problems in this course, adult learners will use the following three subject-specific competencies:

- *Uses strategies to solve situational problems*
- *Uses mathematical reasoning*
- *Communicates by using mathematical language*

The use of effective strategies involves employing rigorous mathematical reasoning and communicating clearly while observing the codes and conventions of mathematical language. Adult learners solve situational problems by using all three subject-specific competencies and other resources.

The following section explains how to use the three subject-specific competencies to solve a situational problem.

PROCESS AND STRATEGIES

To solve a situational problem, adult learners need effective strategies that they can adapt to the situations at hand.

Adult learners solve situational problems using a four-phase process:

- **representation**
- **planning**
- **activation**
- **reflection**

The following table gives an overview of the phase in the problem-solving process, as well as a few examples of strategies adult learners can use in dealing with various situations. These phases are not necessarily carried out in the order indicated above. Adult learners may have to go back and forth among the four phases in order to solve a situational problem.

PROCESS AND STRATEGIES

REPRESENTATION

- Adult learners examine the situational problem to identify the context, the problem and the task to be performed. They use observational and representational strategies that are essential to inductive reasoning.
- They increase their knowledge of mathematical notation and symbols related to functions and inverse functions expressed in the general form.

Examples of strategies	• Writing literal expressions to represent the elements of the situation that seem relevant, thus making it easier to identify a dependency relationship and determine the variables in the situation • Using examples involving numbers, determining the types of relationship that exist between the variables in the situation • Using a logarithmic scale

PLANNING

- In planning their solution, adult learners look for ways of approaching the problem and choose those that seem the most efficient.
- They attempt to extrapolate results using an algebraic rule or a graph, thus expanding their networks of cognitive resources.
- To correctly plan their solution, they decode elements of mathematical language, such as the meaning of the symbols, terms and notation used, as well as the different registers of representation.

Examples of strategies	• Drawing a concept map showing the different steps in the solution • Referring to a list of elements to be considered in consolidating their work plan (e.g. the scale of the axes, the increasing and decreasing intervals, the maximum or the minimum, if any) • Exploring registers of representation that highlight the linear nature of the data

ACTIVATION

- When dealing with a situational problem, adult learners use reasoning to establish structured and functional relationships among different aspects of their knowledge, thus expanding their networks of mathematical cognitive resources.
- They use different strategies by associating pictures, objects or concepts with mathematical terms and symbols and by switching from one register of representation to another.

Examples of strategies	• Changing perspective • Systematically determining the general form of the algebraic rule of a function • Finding combinations in order to determine the rule of a quadratic function • Creating a linear model from a non-linear one by replacing the values of the independent (X) or dependent (Y) variable, or both, with their logarithm (ideally, it is preferable to first recognize the model that seems to best fit the scatter plot, and then verify whether the model is correct)

REFLECTION

- Adult learners use a reflective approach throughout the situational problem and always review the steps in the problem-solving process and the choices made, with a view to validating the solution.
- Through reasoning, they could make conjectures about particular or special cases to validate certain results.
- They use different strategies to make sure that the dependent and independent variables are properly defined, that the axes are correctly scaled, that no unit of measure has been omitted and that the data have been correctly transcribed.

Examples of strategies	• Checking their solution by, for example, making sure that the resulting values satisfy the range of the function, or substituting the values of the variables in the algebraic expression in order to validate a graphical interpolation or extrapolation

CROSS-CURRICULAR COMPETENCIES

Cross-curricular competencies are not developed in a vacuum; they are rooted in situational problems. To varying degrees, the cross-curricular competencies contribute to the development of the subject-specific competencies, and vice versa.

Several cross-curricular competencies can be useful in dealing with the family of learning situations *Relationship between quantities*. Two cross-curricular competencies are considered particularly relevant to this course: *Uses information and communications technologies* and *Uses information*.

Methodological Competencies

Adult learners who wish to compile and analyze data related to a situation may use computer tools such as a spreadsheet program or graphing software. These tools make it easier to produce graphs and to change or work with parameters in order to carry out simulations and extrapolations. Through the competency *Uses information and communications technologies*, adult learners will realize that the ability to master these technologies will make their work considerably more interesting.

Intellectual Competency

The information in studies on financial matters is not always clearly presented in running text or tables according to the rules and conventions of mathematics. Data may be collected through polls or surveys and, in this case, must be organized so that it can be interpreted as accurately as possible in order to provide the information required. Adult learners could therefore learn to use information consisting of raw data. The competency *Uses information* will help them distinguish between data and information, and understand that the proper organization of data makes it possible to correctly interpret a situation.

SUBJECT-SPECIFIC CONTENT

In this course, adult learners use and build on their previously acquired knowledge of arithmetic and algebra. In order to deal effectively with situational problems, they will add to what they have learned by mastering the mathematical knowledge specific to this course.

Prescribed Knowledge

In order to deal effectively with the learning situations in this course, adult learners develop the following three integrative processes:

- **using an algebraic or graphical model to represent a situation**
- **interpolating or extrapolating from a graphical model**
- **using an algebraic or graphical model to generalize a set of situations**

These processes, which are applied in the learning situations in this course, foster the integration of mathematical knowledge and the subject-specific competencies. The learning situations must involve at least one of these integrative processes. However, there must be a sufficient variety of learning situations to cover all three processes.

Mathematical Knowledge	Restrictions and Clarifications
Numerical and algebraic expressions • Real numbers o Powers o Logarithms	An arithmetic approach to exponents and logarithms is promoted. Adult learners manipulate expressions and convert them to the same base (base 10, for use with a calculator) so that the exponents can be compared. If necessary, they use such equivalencies as: • $a^b = c \Leftrightarrow \log_a c = b$ • $\log_a c = \frac{\log_b c}{\log_b a}$
Relation, function and inverse • Solving exponential and logarithmic equations using a change of base, if necessary	Adult learners can represent and write numbers in logarithmic notation by using the following equivalency, if necessary: $\log_a x = n \Leftrightarrow a^n = x$

Mathematical Knowledge	Restrictions and Clarifications
Financial mathematics • Calculating, interpreting and analyzing financial situations	Financial calculations are limited to the following concepts: • simple and compound interest (i) • interest period (n) • discounting (present value – C_0) • compounding (future value – C_n) Compounding (future value) is determined using the following formula: $C_n = C_0(1 + i)^n$ Discounting (present value) is determined using the following formula: $C_0 = C_n(1 + i)^{-n}$ The interest rate (i) is determined using the following formula: $i = \left(\frac{C_n}{C_0}\right)^{1/n} - 1$ The interest period is determined using the following formula: $n = \frac{\log\left(C_n/C_0\right)}{\log(1 + i)}$ Compound interest is presented using graphs or compiled data tables. The following aspects may figure in situations involving personal finances: • types of income (e.g. remuneration, salary, commission, contracts, tips and gratuities) • different types of taxes (e.g. income tax, property tax and withholding tax) • types of financing (e.g. in-store financing, personal loans and mortgages), taking into account the charges associated with each one • cost of services (e.g. telephone, electricity)

Cultural References

Many mathematicians in the late 16th century lamented the fact that scientific progress was significantly slowed down by the need to do long and difficult calculations. It was around this time that the mathematician John Napier invented logarithms. Logarithmic tables and the rules of calculation were later developed to facilitate financial calculations, for example. The invention of logarithms had a significant impact on the structure of mathematics. A logarithmic scale is a practical way to represent numbers on a graph, especially when their orders of magnitude differ greatly. Interestingly, in music theory, logarithms are used to describe music intervals.

In science, logarithms are often used in formulas. An example of this is the natural logarithm, in which the mathematical constant e is the base. The natural logarithm is used in physics to interpret many natural phenomena. The base 10 logarithm is used to model earthquakes, and the binary logarithm (base 2) is commonly used in computer science and information theory. Lastly, it should be remembered that the work of physicist Ludwig Boltzmann on entropy and heat transfer led him to deduce his famous formula, which relates entropy S to the number of possible microstates W, i.e. $S = k \log W$, where k is the Boltzmann constant. This formula is carved on his tombstone.

FAMILY OF LEARNING SITUATIONS

The situations in the family *Relationship between quantities* involve problems that can be solved in part by using an algebraic or graphical model of a function to represent a relationship between quantities. The *Algebraic and Graphical Modelling in a General Context 2* course provides adult learners with an opportunity to learn how to express a connection or a dependency relationship between quantities.

In the situational problems in this course, adult learners become more familiar with the mathematical symbols and notation related to functions and their inverse expressed in their general form, extrapolate results using an algebraic rule or a graph, and use a scale appropriate to the context so that the graph they draw in solving the situational problem makes sense in the context.

BROAD AREAS OF LEARNING

The broad areas of learning deal with major contemporary issues. Ideally, the situations to be studied should be selected in keeping with the educational aims of the broad areas of learning, which provide the situational problems with contexts that make the learning process meaningful. Two broad areas of learning are considered particularly relevant to this course: Environmental Awareness and Consumer Rights and Responsibilities, and Career Planning and Entrepreneurship.

Environmental Awareness and Consumer Rights and Responsibilities

Adult learners interested in natural disasters such as earthquakes could, through a learning situation on this topic, establish a relationship between a logarithmic function and the calculation of the magnitude of an earthquake. They would discover that this data relates to a continuous logarithmic function rather than an ordinary proportional scale. Because of the logarithmic nature of this phenomenon, when the energy generated by an earthquake varies by a factor of 10, this corresponds to a one-unit change in magnitude. For example, an earthquake with a magnitude of seven on the Richter scale is ten times stronger than an earthquake with a magnitude of six. Adult learners could use this situation to become more knowledgeable about their environment and improve their understanding of certain phenomena, which ties in directly with one of the focuses of development of this broad area of learning.

Career Planning and Entrepreneurship

In a learning situation involving financial mathematics, adult learners could be asked to determine an annual rate of interest and, if the initial amount invested is given, the value of a term deposit for varying years of investment and its value ten years later. This situation enables adult learners to use their knowledge of exponential functions to develop a more practical understanding of this type of function, while learning about the principles of saving. In this way, they could develop strategies that will be useful in carrying out a personal plan, which ties in directly with one of the focuses of development of this broad area of learning.

EXAMPLE OF A LEARNING SITUATION

All learning situations and situational problems, regardless of the broad area of learning to which they are related, require the active participation of the adult learner. They provide an opportunity to develop the targeted subject-specific and cross-curricular competencies, to acquire mathematical concepts and to mobilize a variety of useful resources.

The table below presents the elements needed to develop a learning situation or situational problem. It specifies these elements for the situational problem described on the following page.

ELEMENTS NEEDED TO DEVELOP A LEARNING SITUATION OR A SITUATIONAL PROBLEM	
Targeted broad area of learning – Helps contextualize learning and makes it meaningful.	• Environmental Awareness and Consumer Rights and Responsibilities • Career Planning and Entrepreneurship
Prescribed subject-specific competencies – Are developed through the active participation of adult learners.	• Uses strategies to solve situational problems • Uses mathematical reasoning • Communicates by using mathematical language
Prescribed family of learning situations – Consists of real-life situations applicable to a given course. – Helps adult learners acquire mathematical knowledge.	• Relationship between quantities
Targeted cross-curricular competencies – Are developed at the same time and in the same context as the subject-specific competencies.	• Uses information and communications technologies • Uses information
Prescribed essential knowledge – Refers to mathematical knowledge and concepts to be acquired.	• See list

This section provides an example of a situational problem along with possible tasks involved in its mathematical processing. The context can be used as a common thread throughout the learning situation. The learning activities are not spelled out; rather, the focus is on a relevant example of mathematical processing using the four phases in the problem-solving process: representation, planning, activation and reflection. Although not explicitly stated, the elements of the situational problem identified in the previous table, i.e. the broad area of learning, subject-specific competencies, family of learning situations, cross-curricular competencies and essential knowledge, can be discerned, and must form a coherent and meaningful whole.

Teachers may choose to use any of these elements as objects of learning. For instance, learning can focus on actions associated with the phases in the problem-solving process, actions related to the subject-specific or cross-curricular competencies, or actions related to the prescribed knowledge. Teachers can also use the example provided to construct other complex tasks or learning activities related to the mathematical knowledge adult learners must acquire.

Situational problem	**Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the *Relationship between quantities* family of learning situations**
An adult learner wants to find out more about the work of a traffic accident reconstruction expert. She wants to become familiar with the concepts related to this type of reconstruction. In addition to gathering information about a particular event, interpreting physical evidence found at the accident site, taking photographs of the scene and making sketches, such an expert draws on certain mathematical concepts.	**Integrative process:** *Using an algebraic or graphical model to generalize a set of situations* In carrying out the four phases in the problem-solving process, adult learners could: Representation • Select the relevant information (mass and acceleration in this case) and disregard superfluous information (e.g. tire traction, reaction time, type of surface, weather conditions) • Reflect on the need to refer to several similar experiments to be able to come to a generalization Planning • Choose several similar experiments involving acceleration and deceleration • List the elements needed to draw the graph (mass and acceleration in this case)

Situational problem	Examples of possible tasks involved in the mathematical processing of a situational problem belonging to the *Relationship between quantities* family of learning situations	
For example, using data resulting from experiments, the adult learner determines the relationship (rule) between the acceleration (or deceleration) of a vehicle and its mass and whether it is possible to generalize this rule, especially when the initial speed is changed.	Activation	• Make a table of the data related to the situation, taking into account the limitations and precision of the measuring instruments used • For a given initial speed, find the algebraic rule showing the relationship between acceleration and mass • Repeat the operation with different initial speeds • Compare the resulting relationships in order to derive a general rule of correspondence between acceleration and mass (the rule should be valid regardless of the initial speed)
	Reflection	• Suggest probable or plausible reasons that the equation is not perfectly consistent with the data analyzed (e.g. human error, measurement errors, limitations of the instruments used to take the measurements)

END-OF-COURSE OUTCOMES

To solve situational problems in the family of learning situations *Relationship between quantities*, adult learners represent a situation, carry out interpolations and extrapolations and use an algebraic or graphical model to generalize a set of situations. To do this, they use the three subject-specific competencies, *Uses strategies to solve situational problems*, *Uses mathematical reasoning* and *Communicates by using mathematical language*.

When representing a situation using an algebraic or graphical model, adult learners describe, symbolize, code, decode, explain or illustrate information contained in a table of values or a graph of an exponential (or logarithmic) equation. They combine different registers of representation as needed to produce a message in accordance with the notation, rules and conventions of mathematical language. They use problem-solving strategies to make comparisons, propose corrections, present favourable or optimal solutions, or issue recommendations. They formulate constructive criticism and make informed decisions in accordance with the conclusions they have drawn from the mathematical treatment of the situational problem.

In interpolating or extrapolating results from an algebraic or graphical model, adult learners use their knowledge of different types of functions and strategies, combining reasoning and creativity to overcome obstacles and make decisions. They use structured deductive reasoning in order to extract the linear quality of the data in situations that involve an exponential or logarithmic component.

To generalize a set of situations using an algebraic or graphical model, adult learners specify the purpose of their communication and switch from one register to another as needed. They demonstrate their understanding of the mathematical concepts in question using a wide range of communication strategies, which enables them to regulate the transmission of a message based on the specific reactions of the audience or to take new requirements into account. They learn and correctly use language that appropriately combines common mathematical terms. Lastly, by analyzing a process of generalization, they are able to induce laws and formulas specific to the world of finance, for example.

Throughout the problem-solving process, adult learners make an effort to apply their mathematical knowledge (basic second-degree polynomial functions and exponential functions). They construct new knowledge by induction and generalize a set of situations that they validate using different sources so that they can improve their mathematical literacy. In addition, they do not hesitate to ask for help when they encounter difficulties.

EVALUATION CRITERIA FOR THE COMPETENCIES TARGETED BY THE COURSE

Uses strategies to solve situational problems

- *Indication (oral or written) that the situational problem has been understood*
- *Application of strategies and appropriate mathematical knowledge*
- *Development of an appropriate solution**
- *Appropriate validation of the steps** in the solution*

* The solution includes a procedure, strategies and a final answer.
** The mathematical model, operations, properties or relations involved.

Uses mathematical reasoning

- *Formulation of a conjecture suited to the situation*
- *Correct use of appropriate mathematical concepts and processes*
- *Proper implementation of mathematical reasoning suited to the situation*
- *Proper organization of the steps in an appropriate procedure*
- *Correct justification of the steps in an appropriate procedure*

Communicates by using mathematical language

- *Correct interpretation of a mathematical message*
- *Production of a message in keeping with the terminology, rules and conventions of mathematics, and suited to the context*